

# STUDYING THE AUTOMORPHISM GROUP OF THE HIGMAN-THOMPSON GROUP $G_{2,1}$ USING UNIVERSAL ALGEBRA

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ABSTRACT. We view Thompson's group  $V$  as Graham Higman did, the automorphism group (called  $G_{2,1}$ ) of a certain free algebra coming from a universal algebraic structure. We use this point of view to study the automorphism group of  $G_{2,1}$ ,  $Aut(G_{2,1})$ .

## 1. INTRODUCTION

The world of Thompson's group  $V$  is vast and there are many ways to study the group, from the point of view of dynamics [2, 3, 5, 12, 15], as permutations of dyadic partitions of the unit interval [6] and even diagrams [1, 12]. Many more references could be given to these approaches. Brin in [5] studied the automorphism group for Thompson's group  $F$  and  $T$  by using dynamics. Here, we will take advantage of the work of Graham Higman in [9] to study  $Aut(G_{2,1})$ . We find the following.

**Theorem 1.1.**

$$1 \rightarrow G_{2,1} \rightarrow Aut(G_{2,1}) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1.$$

The author has been informed that this work was originally carried out by Collin Bleak, Daniel Lanoue and Yonah Maissel [4] using the dynamical point of view and the theory of transducers described by Grigorchuk. They give the general result

$$Aut(G_{n,1}) \cong Inn(G_{n,1}) \rtimes \Sigma_n$$

for  $\Sigma_n$  the symmetric group on  $n$  symbols,  $n \geq 2$ . The methods and techniques involved are very far removed from this papers.

**1.1. Universal algebras.** We will now describe some useful background material on universal algebra, which can be found in [10, Chapter 1, §1.3].

Given a set  $S$ , we say a mapping  $f: S^n \rightarrow S$  is an  $n$ -ary algebraic operation on  $S$ .

**Definition 1.2.** We define an *algebraic system*  $\mathcal{A} = (S, f_i : i \in I)$  to consist of a set  $S$  together with a set of mappings  $\{f_i : i \in I\}$  where each  $f_i$  is an  $n_i$ -ary algebraic operation on  $S$  for some  $n_i \in \mathbb{N}$ . We call  $\langle n_i : i \in I \rangle$  the signature of  $\mathcal{A}$ .

Homomorphisms of algebraic systems are mappings that preserve all operations. Thus, as a bijective homomorphism is an isomorphism then we see that an isomorphism of an algebraic system onto itself is an automorphism of the algebraic system.

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We say that a set  $X \subseteq \mathcal{A}$  is a set of free generators for the algebraic system  $\mathcal{A}$  if  $X$  generates  $\mathcal{A}$  (written  $\mathcal{A} = \langle X \rangle$ ) and if for  $X = \{x_j | j \in J\}$  any elements  $y_j, j \in J$ , in any algebraic system  $\mathcal{B}$  of the same signature the mapping defined by  $x_j \mapsto y_j$  extends to a homomorphism of  $\mathcal{A}$  to  $\mathcal{B}$ . This homomorphism is unique since  $\mathcal{A} = \langle X \rangle$ . We then say that  $\mathcal{A}$  is free on the free set of generators  $X$ . If  $\mathcal{A}'$  is also free on the generators  $x'_j, j \in J$ , then the composition of the two homomorphisms that extend the mappings  $x_j \mapsto x'_j$  and  $x'_j \mapsto x_j$  act as the identity on the  $x'_j$ 's and so both homomorphisms are bijections.

A free system  $\mathcal{A}$  can be constructed as the set of all formal expressions in the  $x_j$ 's under arbitrarily repeated applications of the  $n_i$ -ary operations  $f_i$ .

**Definition 1.3.** A variety  $\mathcal{V}$  is the class of all algebraic systems of a fixed signature defined by a given set of laws that are expressions in the operations of the algebraic systems.

We can therefore think about an automorphism of a variety  $\mathcal{V}$  as a endofunctor of that variety which has an "inverse". It therefore has to preserve all aspects of the algebraic structure given by the set of laws.

**Example 1.4.** The class of all groups is the variety defined by the group laws.

An algebraic system  $\mathcal{A} = \langle X \rangle$  is free in the variety  $\mathcal{V}$  on the set of free generators  $X$  if for any element  $y_j, j \in J$ , in any algebraic system  $\mathcal{B} \in \mathcal{V}$  the mappings  $x_j \mapsto y_j$  extends to a unique homomorphism of  $\mathcal{A}$  into  $\mathcal{B}$ .

**1.2. The variety  $\mathcal{V}_2$  and the free algebra  $V_{2,1}$ .** We can formulate in full generality (as Higman did in [9]) in terms of universal algebra for an algebraic system consisting of a set  $S$  and a one-to-one mapping from  $S$  to its  $n^{\text{th}}$  cartesian product  $S^n$ .

The mapping  $S \rightarrow S^n$  may be described using  $n$  mappings  $\alpha_i : S \rightarrow S$  for  $i = 1, \dots, n$ , where

$$(1) \quad a \mapsto (a\alpha_1, \dots, a\alpha_n).$$

We view these mappings  $\alpha_i$  as unary operations on  $S$ . As the mapping  $S \rightarrow S^n$  is one-to-one and onto, it has an inverse  $\lambda : S^n \rightarrow S$  which we may view as an  $n$ -ary operation on  $S$ .

The operations  $\alpha_i$  and  $\lambda$  must satisfy the following laws,

$$(2) \quad a\alpha_1 \dots a\alpha_n \lambda = a$$

$$(3) \quad a_1 \dots a_n \lambda \alpha_i = a_i \text{ for } i = 1, \dots, n$$

for all  $a, a_1, \dots, a_n \in S$ . Conversely, if  $S$  is any set on which the operations  $\alpha_i$  and  $\lambda$  are defined and satisfy (2) and (3), the mapping (1) determines a one-to-one correspondence  $S \rightarrow S^n$ .

Therefore, we introduce the variety  $\mathcal{V}_2$  of universal algebras with  $n$  unary operations  $\alpha_1, \dots, \alpha_n$  and one  $n$ -ary operation  $\lambda$  satisfying the laws (2) and (3).

For each pair of integers  $n \geq 2$ ,  $r \geq 1$ , let  $V_{n,r}$  be the free algebra of  $\mathcal{V}_n$  with  $r$  free generators. Any free algebra in  $\mathcal{V}_n$  that is freely generated by a set  $X$  is also freely generated by the set  $(X \setminus \{x\}) \cup \{x\alpha_1, \dots, x\alpha_n\}$  for each  $x \in X$ .

For  $\mathcal{V}_2$ , the laws of (2) and (3) become

$$(4) \quad a\alpha_1 a\alpha_2 \lambda = a \text{ and } a_1 a_2 \lambda \alpha_i = a_i,$$

for  $i = 1, 2$  and we also have the induced bijection  $S \rightarrow S^2$ .

Let  $V_{2,1}$  be the free algebra of  $\mathcal{V}_2$  with a single free generator. It was shown in [9] that  $V_{2,r} \cong V_{2,s}$  for all  $r, s \geq 1$ .

We think of the set  $S$  associated to the algebraic system  $V_{2,1}$  as the set of standard forms defined below. Therefore,  $V_{2,1}$  is isomorphic to the algebraic system  $(S, \alpha_1, \alpha_2, \lambda)$  satisfying the laws given by (4).

From [9], we can define the finitely presented infinite simple group  $G_{2,1}$  to be the automorphism group of the free algebra  $V_{2,1}$ . That is,  $G_{2,1}$  is isomorphic to the automorphism group of the algebraic system  $(S, \alpha_1, \alpha_2, \lambda)$ .

We would like to study the automorphism group of  $G_{2,1}$ ,  $Aut(G_{2,1})$ , by looking at automorphisms of the abstract algebraic structure of the variety  $\mathcal{V}_2$  in which  $V_{2,1}$  is an element.

**1.3. The Higman-Thompson group  $G_{2,1}$ .** We recall the definition of the free algebra  $V_{2,1}$  defined by the set  $S$  (described below) and the operations on the set  $S$ ; two unary operations  $\alpha_1, \alpha_2$  and one binary operation  $\lambda$ . This material can be found in [9, Section 2 & 4].

**Definition 1.5.** The set  $S$  of strings in *standard form* is defined as the smallest subset of  $\{x, \alpha_1, \alpha_2, \lambda\}^*$  as follows (where  $A = \{\alpha_1, \alpha_2\}$ ):

- $x \in S$ ,
- $x\Gamma \in S$  for  $\Gamma \in \langle A \rangle$ ,
- if  $w_1, w_2 \in S$  and there is no  $u \in S$  with  $w_i = u\alpha_i$  for  $i = 1, 2$  then  $w_1 w_2 \lambda \in S$ .

We define unary operations  $\alpha_1, \alpha_2$  and a binary operation  $\lambda$  on the set  $S$  of standard forms by:

- $(x\Gamma)\alpha_i = x\Gamma\alpha_i$ ,
- $(w_1, w_2)\lambda = w_1 w_2 \lambda$  unless there exists  $u \in S$  with  $w_i = u\alpha_i$  for  $i = 1, 2$  then  $(u\alpha_1, u\alpha_2)\lambda = u$ ,
- $((w_1, w_2)\lambda)\alpha_i = w_i$ .

*Remark 1.6.* We write  $\langle A \rangle$  to denote the free monoid generated by  $\alpha_1, \alpha_2$ . The singleton set  $\{x\}$  will be denoted by  $\mathbf{x}$ .

For a subset  $X$  of the algebra  $V_{2,1}$ , we shall write  $X\langle A \rangle$  for the  $A$ -subalgebra of  $V_{2,1}$  generated by elements of  $X$  under the application of the operations  $\alpha_i$  for  $i = 1, 2$ , and  $X\langle \lambda \rangle$  for the  $\lambda$ -subalgebra generated by elements of  $X$  under the application of the operation  $\lambda$ . It is possible to see that  $V_{2,1} = \mathbf{x}\langle A \rangle\langle \lambda \rangle$ .

**Lemma 1.7.** [9, Lemma 2.3 (i),(ii)] *Suppose that  $X$  is a free generating set for  $V_{2,1}$  and that  $y, y_1, y_2$  are distinct elements of  $X$ . Then*

(1)

$$(X \setminus \{y\}) \cup \{z \mid z = (y)\alpha_i, i = 1, 2\}$$

is also a free generating set. Note that  $(y)\alpha_i = y\alpha_i$  for  $i = 1, 2$ . We call this new generating set a single expansion of  $X$ ; a finite composition of  $d$  single expansions is called a  $d$ -fold expansion of  $X$ .

(2)

$$(X \setminus \{y_1, y_2\}) \cup \{(y_1, y_2)\lambda\}$$

is also a free generating set. This is called a single contraction of  $X$ ; a finite composition of single contractions is called a contraction.

Higman proved that any expansion of  $X$  freely generates  $V_{2,1}$  and that any single contraction of  $X$  freely generates  $V_{2,1}$ . This implies that  $\mathbf{x}$  freely generates the algebra  $V_{2,1}$ .

**Definition 1.8.** We define a subset of  $\mathbf{x}\langle A \rangle$  to be called a *basis* of  $\mathbf{x}\langle A \rangle$  if it is a  $d$ -fold expansion of  $\mathbf{x}$  for some  $d \in \mathbb{N}$ . We then say that this basis is a finite free basis of  $\mathbf{x}\langle A \rangle$  of size  $d$ .

Let  $G_{2,1}$  be the automorphism group of the free algebra  $V_{2,1}$ , which is freely generated by  $\mathbf{x}$ . Let  $Y, Z$  be finite free bases of  $\mathbf{x}\langle A \rangle$  of size  $d$ . Then any map  $\psi : Y \rightarrow Z$  induces an automorphism of  $V_{2,1}$ .

**Lemma 1.9.** [9, Lemma 4.1] *If  $\{\psi_1, \dots, \psi_k\}$  is a finite subset of  $G_{2,1}$  then there is a unique minimal basis  $Y$  of  $\mathbf{x}\langle A \rangle$  such that  $Y\psi_i \subseteq \mathbf{x}\langle A \rangle$ , for  $i = 1, \dots, k$ . Any other basis of  $\mathbf{x}\langle A \rangle$  with this property is an expansion of  $Y$ .*

Therefore, all automorphisms can be realized by a bijective map between bases of the subalgebra  $\mathbf{x}\langle A \rangle$ .

**1.4. Automorphism groups of categories of free algebras of varieties.** From [11], we see that many people have studied the automorphism groups of free objects  $F$  in a certain variety  $\mathcal{V}$ . Let  $\mathcal{V}^\circ$  be the category of all free (in the variety  $\mathcal{V}$ ) algebras  $F = F(X)$ , where  $X$  is finite and considered as a subset of a universal infinite set  $X^\circ$ .

The idea from [11] was to study  $Aut(\mathcal{V}^\circ)$  by studying the automorphisms of the semigroup of endomorphisms of  $F$ ,  $End(F)$ , for  $W \in Ob(\mathcal{V}^\circ)$ . There is an embedding  $Aut(F) \rightarrow Aut(End(F))$ . The image of  $Aut(F)$  is the group of inner automorphisms of the semigroup  $End(F)$ .

*Remark 1.10.* It is worth noting that  $Aut(F) \subset End(F)$ , with a group structure.

One question stated in [11] is; "what do the towers of automorphisms of free objects look like?" Let  $W$  be a free object and consider the tower of groups,  $Aut(F), Aut^2(F), \dots, Aut^t(F)$ . The minimal  $t$  such that all automorphisms of  $Aut^t(W)$  are inner automorphism is known as the height of the tower. For Free groups, it was shown by [7] that  $Aut(F) \cong Aut^2(F) = Aut(Aut(F))$ .

**Definition 1.11.** [14, p412] A group  $G$  is said to be *complete* if its center and outer automorphism group are both trivial.

**Lemma 1.12.** *The automorphism group of the Higman-Thompson group  $G_{2,1}$  is complete.*

*Proof.* By [14, Theorem 13.5.10 (Burnside)], as  $G_{2,1}$  is a non-abelian infinite simple group,  $Aut(G_{2,1})$  is a complete group.  $\square$

There is an embedding  $\tau_W : Aut(End(F)) \rightarrow Aut(Aut(F))$  and some work has been done on investigating the kernel and image of this map, see for example [8].

Further work on the general case of automorphisms of Universal algebra is presented in [13], focusing on which automorphisms of the variety are inner.

We now briefly outline our approach to studying  $Aut(G_{2,1})$ . Our free algebra is  $V_{2,1}$  and we look at the automorphisms of  $V_{2,1}$ . From [16, Section 5], if all our automorphisms of  $Aut(V_{2,1})$  are quasi-inner then we can study the  $Out(Aut(V_{2,1}))$  by looking at the derived operations (which we will define shortly). Since  $Aut(V_{2,1}) = G_{2,1}$  a non-abelian infinite simple group then  $Inn(G_{2,1}) \cong G_{2,1}$ .

In the work of [16], a general method for looking at automorphisms of  $End(F)$  is given when these automorphisms are defined to be quasi-inner, which we now define below.

**Definition 1.13.** [16, Definition 1] Let  $F$  be a free algebra. An automorphism  $\psi$  of  $End(F)$  is called quasi-inner if there exists a bijection  $s$  of the set  $F$  such that  $\psi(v) = s \circ v \circ s^{-1}$  for every endomorphism  $v$  of  $F$ .

We now look more at [16] and let  $P$  be the subgroup of  $Aut(F)$  consisting of all automorphisms  $\gamma$  of  $F$  such that the restriction of  $\gamma$  to  $X$  (the basis) is a permutation of  $X$  i.e.  $X\gamma = X$ .

**Theorem 1.14.** [16, Theorem 3] *For any automorphism  $\Phi$  of  $End(F)$  the following conditions are equivalent,*

- (1)  $\Phi$  is quasi-inner;
- (2)  $\Phi$  acts on the group  $P$  in the inner way, that is, there exists an automorphism  $\sigma$  of  $F$  such that  $\Phi(\gamma) = \sigma \circ \gamma \circ \sigma^{-1}$  for every automorphism  $\gamma$  from  $P$ ;
- (3) there exists a basis  $\tilde{X}$  of  $F$  with the same number  $n$  of elements such that the group  $\Phi(P)$  leaves fixed  $\tilde{X}$  for all  $\gamma \in P$ .

As [16, Section 5] stated, Theorem 1.14 reduced the problem of studying automorphisms of  $Aut(F)$  to studying automorphisms  $\Phi$  which preserve  $P$ . That is,  $\Phi$  preserves all automorphisms from  $P$ .

Let  $\tilde{X}$  be a basis of  $F(X)$  satisfying  $\tilde{X}\gamma = \tilde{X}$  for all  $\gamma \in P$ . Let  $\sigma$  be the automorphism which maps  $X \rightarrow \tilde{X}$ . Set  $\Psi(v) = \sigma^{-1} \circ \Phi(v) \circ \sigma$  for every  $v \in End(F)$ .  $\Psi$  is obviously an automorphism of  $End(F)$  and  $\Phi(\gamma) \in P$  for all  $\gamma$ .

Now, consider the set  $M$  of all endomorphisms  $v$  of  $F$  such that  $v \circ \gamma = v$  for all  $\gamma \in P$ . By the above assumptions,  $\Phi(M) = M$ . Thus  $\Phi(M)$  determines a permutation of  $M$ . According to the definition,  $v \in M$  if and only if it fixes  $X$ . That is, there is a

correspondence between elements of  $F$  and elements of  $M$ , namely, for every  $a \in F$  there exists a unique endomorphism  $\nu_a$  from  $M$  such that  $\nu_a(x) = a$  for every  $x \in X$ .

The permutation that realizes  $\Phi$  on  $M$  determines a permutation  $s$  of  $F$  in the following way:

$$s(a) = \Phi(\nu_a)(x) \quad \forall a \in F.$$

Let  $\mu$  be an endomorphism of  $F$ . Since  $\mu \circ \nu_a = \nu_{\mu(a)}$  we obtain, for every  $a \in F$ ,

$$\Phi(\mu) \circ \nu_{s(a)} = \nu_{s(\mu(a))}$$

where

$$\Phi(\mu) \circ s = s \circ \mu$$

and so

$$\Phi(\mu) = s \circ \mu \circ s^{-1}.$$

**Definition 1.15.** Let  $\Phi$  be an automorphism which preserves the subgroup  $P$ . Thus, by Theorem 1.14,  $\Phi(\mu) = s \circ \mu \circ s^{-1}$  for all  $\mu \in \text{End}(F)$  and  $s(a) = \Phi(\nu_a)(x)$  for all  $a \in F$  for  $\nu_a$  an endomorphism of the basis  $X$ . We call the permutation defined in such a way the main permutation corresponding to  $\Phi$ .

From Theorem 1.14 part (3),  $s(X)$  is also a basis of  $F$  and hence there is an automorphism  $\sigma$  coinciding with  $s$  on  $X$ , giving  $\Phi(\gamma) = \sigma \circ \gamma \circ \sigma^{-1}$  for every  $\gamma \in P$ . Consider a second automorphism  $\Psi(\mu) = \sigma \circ \mu \circ \sigma^{-1}$ . Since  $\Psi|_P = \Phi$ ,  $\Psi$  preserves the group  $P$ .

If we take  $\Gamma = \Psi^{-1} \circ \Phi$ , this acts as the identity on  $P$ . We have,

$$s_\Gamma(x) = \Gamma(\nu_x)(x) = \sigma^{-1} \circ s \circ \nu_x \circ s^{-1} \circ \sigma(x) = \sigma^{-1} \circ s(x) = x.$$

As [16, Section 5] states, this fact allows to reduce the problem of studying  $s$  when  $s$  preserves all the basis elements.

So,  $\Phi$  preserves all automorphisms from  $P$  and its main permutation  $s$  preserves all elements from  $X$ .

Let  $\omega \in \Omega$  be a  $k$ -ary operation in the algebra  $F$ , where  $k \leq m$  ( $|X| = m$ ). Then, the element  $s((x_1, \dots, x_k)\omega)$  determines a new operation  $\omega^*$  in  $F$ . That is, the main permutation determines a derived operation in the same variety.

Since, for every basic operation  $\omega \in \Omega$  we have a new operation  $\omega^*$  of the same arity, the new algebra  $F^*$  is defined of the same type. Such an algebra is a derived algebra.

**Theorem 1.16.** [16, Theorem 6] *The main permutation  $s$  is an isomorphism between  $w$  and  $w^*$ .*

**Corollary 1.17.** [16, Corollary 3] *The main permutation  $s$  is an isomorphism of  $F$  and  $F^*$ .*

The derivative algebra  $F^*$  has the same free basis  $X$ . If we find all such structures, we know all possible permutations  $s$  which determine automorphisms  $\Phi$  of  $\text{Aut}(F)$  which are not inner.

**Lemma 1.18.** *The subgroup  $P$  of  $G_{2,1}$  consisting of all automorphisms of  $V_{2,1}$  such that the restriction to  $\mathbf{x}$  is a permutation of  $\mathbf{x}$  is the identity subgroup.*

*Hence, all automorphisms  $\Phi$  of  $G_{2,1}$  are quasi-inner.*

*Proof.* By definition, elements of  $G_{2,1}$  are bijections  $X \rightarrow Y$  between any two bases  $X, Y$  of the same cardinality. As  $X, Y$  are bases if and only if they are expansions or contractions of  $x$ , then the subgroup of automorphisms  $P$  of  $G_{2,1}$  such that the restriction to  $x$  is a permutation of  $x$ , is the subgroup consisting of just the identity automorphism.

As the subgroup  $P \cong \{Id_{G_{2,1}}\}$ , then all automorphism  $\Phi$  of  $G_{2,1} \subset End(V_{2,1})$  act in the inner way on  $P$  as it is just the identity in  $G_{2,1}$ . Therefore, by Theorem 1.14, all automorphisms of  $G_{2,1}$  are quasi-inner.  $\square$

*Remark 1.19.* We have explicitly stated the above result, even though it follows from [16, Section 3] that for all monogenic free algebras ( $|X| = 1$ )  $F(X)$ , all automorphisms of  $End(F)$  are quasi-inner and thus all automorphisms of  $Aut(F)$  are quasi-inner.

## 2. DERIVED OPERATIONS FOR THE FREE ALGEBRA $V_{2,1}$ IN THE VARIETY $\mathcal{V}_2$

Although "derived" operations are never formally defined in [9], one can understand what is meant by the following example from [9, Section 3].

**Definition 2.1.** Let  $\mathcal{V}$  be a variety of free algebras with  $\mathcal{F}$  the set of operations. A derived operation is an operation which can be constructed from the set  $\mathcal{F}$ .

We see from [9, Section 3], when looking for algebras of  $\mathcal{V}_n$  as algebras of  $\mathcal{V}_N$  for  $N > n$  it makes sense to consider "derived" operations from the standard ones  $\alpha_1, \dots, \alpha_n, \lambda$ .

Let  $S$  be a set of standard forms. Consider the map  $\lambda : S^2 \rightarrow S$ , we can use this map to define a one-to-one correspondence between  $S$  and  $S^3$ .

Suppose that under  $\lambda : S^2 \rightarrow S$  we associate  $a \in S$  with  $(b, c) \in S^2$  and we associate  $b, c$  to  $(d, e)$  and  $(f, g)$  respectively. Then, either of the mappings  $a \mapsto (b, f, g)$  or  $a \mapsto (d, e, c)$  determines a bijective map  $S \rightarrow S^3$ .

That is, we can view a free algebra in  $\mathcal{V}_2$  as a free algebra in  $\mathcal{V}_3$  (in two different ways). One can thus "derive" the unary operations  $\beta_1, \beta_2, \beta_3$  and a 3-ary operation  $\mu$  by,

$$\beta_1 := \alpha_1, \beta_2 := \alpha_2\alpha_1, \beta_3 := \alpha_2\alpha_2 \text{ and } (\cdot, \cdot, \cdot)\mu := (\cdot, (\cdot, \cdot)\lambda)\lambda$$

or by

$$\beta_1 := \alpha_1\alpha_1, \beta_2 := \alpha_1\alpha_2, \beta_3 := \alpha_2 \text{ and } (\cdot, \cdot, \cdot)\mu := ((\cdot, \cdot)\lambda, \cdot)\lambda.$$

In both cases this makes the free algebra in  $\mathcal{V}_2$  into a free algebra in  $\mathcal{V}_3$  as the laws,

$$a\beta_1a\beta_2a\beta_3 = a \text{ and } (a_1, a_2, a_3)\mu\beta_i = a_i \text{ for } i = 1, 2, 3,$$

with  $a, a_1, a_2, a_3 \in S$  are a consequence of the laws of  $\mathcal{V}_2$ .

However, if we take  $n = N$  then we see by the proof of [9, Lemma 3.1] that any derived unary operations are given by a permutation of the unary operations  $\alpha_1, \dots, \alpha_n$ . We thus get the following corollary for free algebras in the variety  $\mathcal{V}_n$ .

**Corollary 2.2.** *Let  $V_{n,r}$  be a free algebra in the variety  $\mathcal{V}_n$ . Then, the set of all derived operation of the unary operations in the variety  $\mathcal{V}_n$  is in one-to-one correspondence with the set of elements of  $Sym(n)$  (the symmetric group on  $n$  letters).*

*Remark 2.3.* This begins to explain the result from [4], which has  $Sym(n)$  as the group of outer automorphisms from the point of view of this section.

**Lemma 2.4.** *Let  $F$  be a free algebra. If all automorphisms  $\psi$  of  $\text{Aut}(F)$  are quasi-inner, then the outer automorphism group is isomorphic to the group of permutations of the defining operations  $\Omega$  which preserve the laws  $\Sigma$ .*

*Proof.* From [16] (see Theorem 1.16 and Corollary 1.17), if we consider all quasi-inner automorphisms, then those that are not inner automorphisms come from a permutation of the set of operations  $\Omega$  which act on the free  $\Omega$ -algebra which respect the laws for the variety in which the free algebra is defined *i.e.* non-inner automorphisms are defined by the main permutation.

Thus, the outer automorphism group is isomorphic to the group of permutations of the defining operations  $\Omega$  which preserve the laws  $\Sigma$ .  $\square$

**Lemma 2.5.**

$$\text{Out}(G_{2,1}) \cong \mathbb{Z}_2.$$

*Proof.* By definition, the algebra  $V_{2,1}$  has the following laws and operations:

- Operations  $\lambda, \alpha_1, \alpha_2$ ;
- Laws

$$((a)\alpha_1, (a)\alpha_2)\lambda = a \text{ and } ((a_1, a_2)\lambda)\alpha_i = a_i$$

for  $i = 1, 2$  and for any  $a_1, a_2, a \in V_{2,1}$ .

The set of permutations of the operations, which respect the laws, can be seen in the following way; we can permute the lower index of  $\alpha_i$  by an element  $\sigma \in \mathbb{Z}_2$ , and this respects the laws for the variety  $\mathcal{V}_2$ :

$$\alpha_{\sigma(1)} = \alpha_2, \alpha_{\sigma(2)} = \alpha_1.$$

Thus, we see that the permutation group associated to the derived operations is  $\mathbb{Z}_2$  and hence, by Lemma 2.4, we have  $\text{Out}(G_{2,1}) \cong \mathbb{Z}_2$ .  $\square$

**PROOF OF THEOREM 1.1:** As  $G_{2,1}$  is a non-abelian finitely presented infinite simple group, the group of inner automorphisms  $\text{Inn}(G_{2,1})$  is isomorphic to the group  $G_{2,1}$ .

From Lemma 2.5, we see that the group  $\text{Out}(G_{2,1}) \cong \mathbb{Z}_2$  and the proof is complete.  $\square$

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