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An algebraic approach to centralizers and conjugacy problems in the Higman-Thompson group $G_{2,1}$

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Richard Thompson's group V can be thought of as a group of automorphisms of a free algebra, which was described by Graham Higman in 1974. Here, we will review and use the original techniques developed by Higman to describe the centralizers for elements of this group. We also consider conjugacy problems in the group $G_{2,1}$, after a review to Graham Higman's original solution to the conjugacy problem.

Keywords: Higman-Thompson group $G_{2,1}$; Thompson's group V; Centralizers; Universal algebra, conjugacy problems.

1. Introduction

In 1974 Graham Higman [9] constructed an infinite series of groups $G_{n,r}$ (for $n \geq 2$ and $r \in \mathbb{N}$) such that all the groups $G_{n,r}$ were finitely presented infinite groups and each $G_{n,r}$ was either simple (for n even) or contained a normal subgroup which was simple (for n odd).

The construction Higman used was based on a report of F. Galvin and R. Thompson [unpublished] on the work of B. Jónsson and A. Tarski [10]. There, Thompson observed that the automorphism group of the Jónsson-Tarski algebra of type $\langle 2, 1, 1\rangle$ was isomorphic to V, that is $G_{2,1}$ in Higman's notation.

Bleak et al. [3] have described the structure of the centralizers for elements of $G_{2,1}$ using the dynamical information gained from the groups action on the Cantor set and the definitions and lemmas of Brin [5] on revealing tree pairs. In the present article, we only use the techniques originally described by Graham Higman in [9] to present an alternative approach to describe the structure of the centralizers for elements of $G_{2,1}$ in a hope to renew interest in this point of view of Thompson's groups. For a review of the concepts of finite presentability and simplicity of $G_{2,1}$, the survey article of Cannon, Floyd and Parry [6] is still the preferred text.

We will now describe some useful background material for the work in [9], which can be found in [12, Chapter 1, §1.3] . This material is standard in the field of Universal Algebra (a good general reference for this material is [7]).

Given a set S, we say a mapping such that $f\colon S^n \to S$ is an n-ary algebraic operation on S.

Definition 1. *We define an* algebraic system $\mathcal{A} = (S, f_i : i \in I)$ *to consist of a set* S *together with a set of mappings* $\{f_i : i \in I\}$ *such that each* f_i *is an* n_i*-ary algebraic operation on* S *for some* $n_i \in \mathbb{N}$*. We call* $\langle n_i : i \in I\rangle$ *the signature of* $\mathcal{A}$*.*

Homomorphisms of algebraic systems are mappings that preserve all operations. Thus, as a bijective homomorphism is an isomorphism then we see that an isomorphism of an algebraic system onto itself is an automorphism of the algebraic system.

Definition 2. *Let* $\mathcal{A} = (S, f_i : i \in I)$ *be an algebraic system. We say that the set* $X \subseteq \mathcal{A}$ *generates* $\mathcal{A}$ *if every element of* S *can be constructed from the set* X *by application of the algebraic operations* f_i *for* $i \in I$*.*

We say that a set $X \subseteq \mathcal{A}$ is a set of free generators for the algebraic system $\mathcal{A}$ if X generates $\mathcal{A}$ (written $\mathcal{A} = \langle X\rangle$) and if for $X = \{x_j | j \in J\}$ any elements y_j, $j \in J$, in any algebraic system $\mathcal{B}$ of the same signature the mapping defined by $x_j \mapsto y_j$ extends to a homomorphism of $\mathcal{A}$ to $\mathcal{B}$. This homomorphism is unique since $\mathcal{A} = \langle X\rangle$. We then say that $\mathcal{A}$ is free on the free set of generators X.

A free system $\mathcal{A}$ can be constructed as the set of all formal expressions in the x_j's under arbitrarily repeated applications of the n_i-ary operations f_i.

Definition 3. *A variety* $\mathcal{V}$ *is the class of all algebraic systems of a fixed signature defined by a given set of laws that are expressions in the operations of the algebraic systems.*

Example 4. The class of all groups is the variety defined by the group laws.

2. $V_{2,1}$ as an Algebra of Standard forms

We shall define $G_{2,1}$ as the group of all automorphisms of an algebraic system $V_{2,1}$, which is a free object in the variety $\mathcal{V}_2$ with signature $\langle 2, 1, 1\rangle$ as defined by Graham Higman in [9].

An element of $\mathcal{V}_2$ consists of a set S, equipped with a pair of unary operations $\boldsymbol{\alpha}_1 : S \to S$ and $\boldsymbol{\alpha}_2 : S \to S$, and a binary operation $\boldsymbol{\lambda} : S \times S \to S$ where the following rules are satisfied

$$((a)\boldsymbol{\alpha}_1, (a)\boldsymbol{\alpha}_2)\boldsymbol{\lambda} = a \text{ and } ((a_1, a_2)\boldsymbol{\lambda})\boldsymbol{\alpha}_i = a_i \tag{1}$$

for $i = 1, 2$ and for any $a_1, a_2, a \in S$.

In order to be able to work with the algebraic structure $V_{2,1}$, we define the set S in the following way.

Remark 5. We remind the reader that we write $\langle A\rangle$ to denote the free monoid generated by α_1, α_2.

Definition 6. *The set* S *of strings in* standard form *is defined as the smallest set of strings over the alphabet* $\{x, \alpha_1, \alpha_2, \lambda\}$ *satisfying:*

- $x \in S$,
- $x\Gamma \in S$ *for* $\Gamma \in \langle A \rangle$,
- *if* $w_1, w_2 \in S$ *and there is no* $u \in S$ *with* $w_i = u\alpha_i$ *for* $i = 1, 2$ *then* $w_1 w_2 \lambda \in S$.

We define unary operations $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ *and a binary operation* $\boldsymbol{\lambda}$ *on the set* S *of standard forms by:*

- *for* $\Gamma \in \langle A \rangle$, $(x\Gamma)\boldsymbol{\alpha}_i = x\Gamma\alpha_i$,
- *for* $w_1, w_2 \in S$, *if there exists* $u \in S$ *such that* $w_i = u\alpha_i$ *for* $i = 1, 2$ *then* $(u\alpha_1, u\alpha_2)\boldsymbol{\lambda} = u$ *otherwise* $(w_1, w_2)\boldsymbol{\lambda} = w_1 w_2 \lambda$,
- $((w_1, w_2)\boldsymbol{\lambda})\boldsymbol{\alpha}_i = w_i$.

Graham Higman in [9, Lemma 2.1] proved that the algebraic structure $(S, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\lambda})$ we have now defined on S is a free object in $\mathcal{V}_2$ called $V_{2,1}$ and is freely genearted by the singleton set $\{x\}$, which we will now denote **x**. We shall use this representation of $V_{2,1}$.

[9, Chapter 2] contains more information on the algebraic system $V_{2,1}$, in fact it contains information on the more general algebraic systems $V_{n,r}$ (which is freely generated by the set X of size r). For the purposes of this article, we only state the results and some of the proofs from [9] that will be essential in understanding the algebraic system $V_{2,1}$.

We start, as Higman did, by creating a subalgebra of $V_{2,1}$ whose elements contain no λs.

Definition 7. *We define the* number of λ *involved in an element of* $V_{2,1}$ *to be the number of* λ *that are in the standard form for that element.*

Definition 8. *For a subset* X *of an algebra* $V_{2,1}$ *and* $A = \{\alpha_1, \alpha_2\}$ *we shall write* $X\langle A \rangle$ *for the* A-subalgebra *of* $V_{2,1}$ *generated by elements of* X *under the application of* $\boldsymbol{\alpha}_i$ *for* $i = 1, 2$, *and* $X\langle \lambda \rangle$ *for the* λ-subalgebra *generated by elements of* X *under the application of* $\boldsymbol{\lambda}$.

Lemma 9. *[9, Lemma 2.2] If a set* X *generates the algebra* $V_{2,1}$ *in* $\mathcal{V}_2$ *then* $V_{2,1} = X\langle A \rangle \langle \lambda \rangle$. *Also, for each* $y \in V_{2,1}$, *the set* $y\langle A \rangle \setminus X\langle A \rangle$ *is finite.*

Proof. [9, Lemma 2.2] First, it is sufficient to consider $V_{2,1}$ free on X. Then $V_{2,1}$ is isomorphic to the algebra of standard forms on X by [9, Lemma 2.1]. So the first part holds by the definition of standard forms, Definition 6. For the last part, it is noted that if $y \in V_{2,1}$ and the number of λ involved in y (in standard form) is m, then we have $y\alpha_{i_1}...\alpha_{i_r} \in X\langle A \rangle$ whenever $r \geq m$. That is, if the words $y\alpha_{i_1}...\alpha_{i_r}$ are sufficiently long the element is pushed into the subalgebra $X\langle A \rangle$. Hence, the only elements of the set difference $y\langle A \rangle \setminus X\langle A \rangle$ are those of the form $y\alpha_{i_1}...\alpha_{i_r}$ with $r < m$, and there are clearly only finitely many in number. □

Lemma 10. *[9, Lemma 2.3 (i),(ii)] Suppose that* X *is a free generating set for* $V_{2,1}$ *and that* y, y_1, y_2 *are distinct elements of* X. *Then*

(1) $$(X \setminus \{y\}) \cup \{y\alpha_1, y\alpha_2\}$$

is also a free generating set. We call this new generating set a single expansion of X; a finite composition of d single expansions is called a d-fold expansion of X.

(2) $$(X \setminus \{y_1, y_2\}) \cup \{(y_1, y_2)\lambda\}$$

is also a free generating set. This is called a single contraction of X; a finite composition of single contractions is called a contraction.

Definition 11. *Let Y be a basis of $V_{2,1}$. We define a proper expansion $\widetilde{Y}$ of Y to be a basis of $V_{2,1}$ which can be reached by d-fold expansion of Y.*

Higman proved that any expansion of X freely generates $V_{2,1}$ and that any single contraction of X freely generates $V_{2,1}$. We therefore see that $\mathbf{x}$ freely generates the free algebra $V_{2,1}$.

Definition 12. *We call a subset Y of $\mathbf{x}\langle A\rangle$ a (finite free)* basis *of $\mathbf{x}\langle A\rangle$ if it is a proper expansion of $\mathbf{x}$* i.e. *a d-fold expansion of $\mathbf{x}$ for some d. The size of the basis Y, denoted $|Y|$, is d.*

In fact, we can say that any finite free basis for $\mathbf{x}\langle A\rangle$ is also a free basis for the free algebra $V_{2,1}$.

Remark 13. We will use X, Y and Z to denote finite free bases for $\mathbf{x}\langle A\rangle$.

Corollary 14. *[9, Corollary 1] Any two finite free bases X, Y of the same algebra have a common expansion Z.*

From Corollary 14 we deduce that if we have a finite collection of finite free bases for the same algebra, $Y_1, ..., Y_n \in V_{2,1}$, then there is a single common expansion of all of them that is also a free basis.

3. Higman-Thompson group $G_{2,1}$

Let $G_{2,1}$ be the automorphism group of the free algebra $V_{2,1}$, which is freely generated by $\mathbf{x}$. Let Y, Z be finite free bases of $\mathbf{x}\langle A\rangle$ of size d. Then any map $\psi : Y \to Z$ induces an *automorphism* of $V_{2,1}$.

We now look to construct a pair of bases for which we can, sensibly, describe an element of $G_{2,1}$.

Lemma 15. *[9, Lemma 4.1] If $\{\psi_1, ..., \psi_k\}$ is a finite subset of $G_{2,1}$ then there is a unique minimal basis Y of $\mathbf{x}\langle A\rangle$ such that $Y\psi_i \subseteq \mathbf{x}\langle A\rangle$, for $i = 1, ..., k$. Any other basis of $\mathbf{x}\langle A\rangle$ with this property is an expansion of Y.*

Proof. [9, Lemma 4.1] The proof uses [9, Lemma 2.4]. There is an expansion Y of $\mathbf{x}$ such that

$$\mathbf{x}\langle A\rangle \cap \bigcap_{i=1}^{k} (\mathbf{x}\psi_i^{-1})\langle A\rangle = Y\langle A\rangle.$$

We have $Y\psi_i \subseteq Y\langle A\rangle\psi_i$ for $i = 1, ..., k$ which is a subset of $(\mathbf{x}\psi_i^{-1})\langle A\rangle\psi_i = \mathbf{x}\langle A\rangle$. Therefore $Y\psi_i \subseteq \mathbf{x}\langle A\rangle$, for $i = 1, ..., k$.

If Z is a basis (that is a d-fold expansion of $\mathbf{x}\langle A\rangle$) with $Z\psi_i \subseteq \mathbf{x}\langle A\rangle$ for $i = 1, ..., k$ then $Z \subseteq Y\langle A\rangle$, and thus Z is an expansion of Y. □

This lemma removes the need to talk about λ for now. We have constructed a basis Y that gives the image of elements of Y under the action of the automorphisms ψ_i for $i = 1, .., k$ in the A-subalgebra we want to look at, $\mathbf{x}\langle A\rangle$. We can now start to look at the orbits of elements of Y (and thus elements of $Y\langle A\rangle$) under the action of powers of elements of $G_{2,1}$.

It is worth explicitly writing the following corollary, which follows from the above lemma.

Corollary 16. *Let Y, Z be finite free bases of $\mathbf{x}\langle A\rangle$ of size d. Then, ψ is an automorphism of the free algebra $V_{2,1}$ if and only if ψ is a bijective map between Y and Z.*

3.1. *Higman's semi-normal form*

This section is based on [9, Section 9], where Higman picks an element ψ of $G_{2,1}$ and constructs a finite basis, Y, making the study of ψ easy (as we can just examine the elements in Y, which is a finite set).

Remark 17. This method is similar to the method of Matt Brin [5] and Bleak et al [3], where revealing tree pairs divulge the dynamical information for a given element of $G_{2,1}$ acting on the Cantor set.

The method is based on consideration of intersections of the orbits of ψ (an automorphism of $V_{2,1}$) with the subalgebra $\mathbf{x}\langle A\rangle$. For any basis Y, such that $Y\psi \subseteq \mathbf{x}\langle A\rangle$ (which exists by Lemma 15), we have elements of Y in one of five different types of orbit under the action of powers of ψ on $V_{2,1}$. We distinguish between the different types of orbit depending on the intersection of the orbit of $y \in Y$ under the action of powers of ψ with the subalgebra $Y\langle A\rangle$. For $y \in Y$, the five different types are:

(1) *Complete infinite orbits.* For any y in such an orbit, $y\psi^i$ belongs to $Y\langle A\rangle$ for all $i \in \mathbb{Z}$, and the elements $y\psi^i$ are all different.
(2) *Complete finite orbits.* For any y in such a orbit, $y\psi^n = y$ for some positive integer n, and $y, y\psi, ..., y\psi^{n-1}$ all belong to $Y\langle A\rangle$.
(3) *Right semi-infinite orbits.* For some y in the orbit, $y\psi^i$ belongs to $Y\langle A\rangle$ for all $i \geq 0$, but $y\psi^{-1}$ does not. The elements $y\psi^i$, $i \geq 0$, are then, of course, necessarily all different.
(4) *Left semi-infinite orbits.* For some y in the orbit, $y\psi^{-i}$ belongs to $Y\langle A\rangle$ for all $i \geq 0$, but $y\psi$ does not. The elements $y\psi^{-i}$, $i \geq 0$, are then, of course, necessarily all different.

(5) *Incomplete orbits.* For some y and some non-negative integer n we have $y, y\psi, ..., y\psi^n$ belonging to $Y\langle A\rangle$ but $y\psi^{-1}$ and $y\psi^{n+1}$ do not.

Remark 18. The elements $y\psi^{-1}$ and $y\psi^{n+1}$ for y in a type (5) orbit under the action of ψ must be contained in the finite set $\mathbf{x}\langle A\rangle \setminus Y\langle A\rangle$. Otherwise, some image of y would be an element of $V_{2,1}$ involving λ's and therefore y would be sitting in an orbit of type (3) or (4).

We can now look at what this means for a basis Y and an element $\psi \in G_{2,1}$ in terms of the different types of orbits.

Lemma 19. *[9, Lemma 9.1] There are only finitely many orbits of types* (3), (4) *and* (5) *intersecting* $Y\langle A\rangle$ *and there are as many of type* (3) *as of type* (4) *under the action of* ψ.

Definition 20. *[9, Section 9] An element ψ of $G_{2,1}$ is in* semi-normal *form with respect to the basis Y if Y intersects no incomplete orbits for ψ* i.e. *no elements of Y are in orbits of type (5).*

We can now define a basis for which $\psi \in G_{2,1}$ is in semi-normal form.

Lemma 21. *[9, Lemma 9.2] For an element ψ of $G_{2,1}$ there exists a basis with respect to which ψ is in semi-normal form.*

This then leads to an examination of the orbits of elements $y\Gamma$ where $y \in Y$ and $\Gamma \in \langle A\rangle$ under the action of powers ψ, where ψ is given in semi-normal form with respect to Y.

Lemma 22. *[9, Lemma 9.3]*

Let $\psi \in G_{2,1}$ be in semi-normal form with respect to the basis Y. Suppose that $y \in Y$, then one of the following holds,
(A) if some $y\Gamma$ is in a complete finite orbit then y itself belongs to a complete finite orbit which consists of elements of Y. In this case we say y is of type (A).
(B) there exists a non-trivial $\Gamma \in \langle A\rangle$ and $n \neq 0$ such that $y\psi^n = y\Gamma$. If $n > 0$ then the orbit containing y is right semi-infinite, if $n < 0$ then the orbit containing y is left semi-infinite. In this case we say y is of type (B).
(C) if $y \in Y$, is not of type (A) nor (B) above, then there exists some $z \in Y$ and non-trivial $\Delta \in \langle A\rangle$ such that $y\psi^i = z\Delta$, where z is of type (B). Then the orbit containing y is infinite. In this case we say y is of type (C).

Thus, this lemma tells us that for ψ given in semi-normal form with respect to the basis Y:

- $y \in Y$ is in a finite orbit of ψ if and only if for all $\Gamma \in \langle A\rangle$, the element $y\Gamma$ is in a finite orbit of ψ in $Y\langle A\rangle$ if and only if y is of type (A);
- $y \in Y$ is in a semi-infinite orbit if and only if there exists some $n \in \mathbb{Z}$ such that $y\psi^n = y\Gamma$ for $\Gamma \in \langle A\rangle$ (with the sign of n determining the type of semi-infinite orbit y is in) if and only if y is of type (B).

We will often refer to elements of type (A), (B) and (C). We now introduce a modified definition from [9, Section 9].

Definition 23. *Let $u \in V_{2,1}$, $\Gamma \in \langle A \rangle$ and $m \in \mathbb{Z}$. We call u a* characteristic element *for ψ^m and Γ a* characteristic multiplier *if $u\psi^m = u\Gamma$.The element and the multiplier are proper if Γ is non-trivial.*

We end this subsection with a result that allows us to determine if an element of $G_{2,1}$ is of infinite order.

Theorem 24. *[9, Theorem 9.4] An element ψ of $G_{2,1}$ (given in semi-normal form with respect to Y) is of infinite order if and only if for some $m \neq 0$, ψ^m has a proper characteristic element* i.e. *$y\psi^m = y\Gamma$ for $y \in Y$ and non-trivial $\Gamma \in \langle A \rangle$.*

Proof. [9, Theorem 9.4] Let ψ be in semi-normal form with respect to the basis Y. If u is a characteristic element for ψ^m with multiplier Γ then $u\psi^{mj} = u\Gamma^j$ for $j \in \mathbb{N}$. When Γ is a proper multiplier, the elements $u\Gamma^j$ are all different as soon as j is large enough to belong to $Y\langle A \rangle$, so that ψ has infinite order.

Conversely, if no ψ^m has a proper characteristic element then Y has no elements of type (B) nor type (C). Thus all elements of Y are of type (A) as ψ is in semi-normal form with respect to the basis Y. Whence ψ is a permutation of Y and thus has finite order. □

3.2. *Quasi-normal forms*

A stronger definition than semi-normal form was introduced in [9, Section 9].

Definition 25. *[9, Section 9] An element ψ of $G_{2,1}$ is in* quasi-normal *form with respect to the basis Y if it is in semi-normal form with respect to Y, but not with respect to any proper contraction of Y.*

We can now look at the orbits of elements $u, v \in Y\langle A \rangle$ under the action of powers of ψ.

Lemma 26. *[9, Lemma 9.6] If ψ is in quasi-normal form with respect to Y, and if $v = u\psi^m$, $m > 0$ and $u, v \in Y\langle A \rangle$ then $u\psi^i$ belongs to $Y\langle A \rangle$ for $i = 1, ..., m-1$.*

Lemma 27. *[9, Lemma 9.7] Given an element $\psi \in G_{2,1}$, (i) we can construct a unique basis with respect to which ψ is in quasi-normal form and (ii) for $u, v \in V_{2,1}$ we can tell whether u, v are in the same orbit of ψ, and if so, what are the integers m for which $u\psi^m = v$.*

Thus, for all $\psi \in G_{2,1}$ we can do the following:

(1) put ψ in quasi-normal form with respect to a basis Y,
(2) for any $u, v \in V_{2,1}$ we can determine if u, v are in the same orbit of ψ and determine which integers m give $v = u\psi^m$ if they are in the same orbit,
(3) if $m > 0$ we have $u\psi^i \in Y\langle A \rangle$ for all $i = 1, ...m-1$.

4. Higman's ψ-admissible subalgebras V_P and V_{RI}

Given $\psi \in G_{2,1}$, we define $V_{P,\psi}$ to be the set of $v \in V_{2,1}$ such that v is in a finite orbit under the action of ψ and $V_{RI,\psi}$ to be the set of $v \in V_{2,1}$ such that v is in an infinite or semi-infinite orbit under the action of ψ . Where there is no ambiguity, we will write V_P for $V_{P,\psi}$ and V_{RI} for $V_{RI,\psi}$. From [9, Theorem 9.5] (which we state below) we see that $V_{P,\psi}$ and $V_{RI,\psi}$ are subalgebras invariant under the action of the automorphism ψ and $V_{2,1}$ is a free product of the subalgebras $V_{P,\psi}$ and $V_{RI,\psi}$. That is, $V_{2,1}$ can be seen as the coproduct of the sub-algebras V_{RI} and V_P in the category of free algebras We will refer to V_P as the periodic subalgebra and V_{RI} as the regular infinite subalgebra.

Theorem 28. *[9, Theorem 9.5] The finitely generated free algebra $V_{2,1}$ is a free product of the ψ-admissible subalgebras V_P and V_{RI}. If $\psi_P = \psi|_{V_P}$ and $\psi_{RI} = \psi|_{V_{RI}}$ then for two automorphisms ψ and φ, ψ is conjugated to φ by $g \in G_{2,1}$ if and only if ψ_P is conjugated to φ_P by a map $g^{(P)}$ and ψ_{RI} is conjugate to φ_{RI} by a map $g^{(RI)}$, where g is given by the composition of maps $g^{(P)}, g^{(RI)}$.*

The second statement in Theorem 28 is used to analyze not only when two elements are conjugate in $G_{2,1}$ (the automorphism group of $V_{2,1}$) but also in the study of centralizers for elements of $G_{2,1}$. Theorem 28 allows us to consider the torsion and regular infinite part of an element of $G_{2,1}$ separately. Hence, we consider the elements that centralize the torsion part and the regular infinite part of an element separately.

Definition 29. *Let ψ be in semi-normal form with respect to a finite basis Y. Then ψ is a* regular infinite *element of $G_{2,1}$ with respect to Y if there exists no $y \in Y$ in a finite orbit.*

Definition 30. *Let ψ be an element of $G_{2,1}$ in semi-normal form with respect to the basis Y. Then ψ is a* periodic element *of $G_{2,1}$ with respect to Y if for all $y \in Y$, y is in a complete finite orbit.*

5. Conjugacy problems

The conjugacy problem is a classic decision problem, first outlined by Dehn over 100 years ago. It asks,

Problem 31. Does there exists a algorithm to decide whether, upon input of the elements $\psi, \varphi \in G$ there exists an element $g \in G$ such that $g^{-1}\psi g = \varphi$?

We can then look at this problem in two ways. Does there exists an algorithm which gives the answer "yes" if ψ is conjugate to φ and "no" if not? If the answer is "yes", can you find a conjugator g? A solution to either the first question or both questions gives a rich source of information about the group in question.

Further, if the answer is "yes", then there are several other conjugacy based decision problems one can ask about a group G. For example the simultaneous and power-conjugacy problems.

5.1. *The conjugacy problem*

The conjugacy problem in Richard Thompson's groups F, T and V is solvable diagrammatically. This method was described by Francesco Matucci [15] using a variation on the methods first used to solve the conjugacy problem for F by Guba and Sapir for diagram groups [8].

The diagrammatic solution to the conjugacy problem is a beautiful unified solution for F, T and V. An effective solution for the simultaneous conjugacy problem for F has been shown by Matucci in [15] and there also exists an effective solution for simultaneous conjugacy problem for T in [15], but it has not been explicitly written.

The conjugacy problem for Thompson's group V was originally solved by Higman in [9, Theorem 9.3], that is using combinatorial methods to provide a solution for the groups $G_{n,r}$ for $n \geq 2$, $r \geq 1$. Salazar-Diaz [19] then used techniques introduced by Brin [5] to provide another solution using revealing pairs tree pairs.

We shall now present a deconstructed solution to the conjugacy problem for the group $G_{2,1}$. The work is based on the proof of [9, Theorem 9.3].

Let ψ and φ be automorphisms of $V_{2,1}$ in quasi-normal form with respect to the bases X and Y respectively. We will want to refer to the algebra $V_{2,1}$ freely generated by the basis X and freely generated by the basis Y, therefore we will refer to $X\langle A\rangle\langle\lambda\rangle$ and $Y\langle A\rangle\langle\lambda\rangle$ as such algebras (instead of writing $V_{2,1}$). By Theorem 28, ψ is conjugate to φ if and only if ψ_P is conjugate to φ_P and ψ_{RI} is conjugate to φ_{RI}.

We can thus deconstruct the proof of the conjugacy problem into two parts. We start with those elements that have $V_{RI,\psi} = V_{RI,\phi} = \emptyset$.

Definition 32. *Let ψ be a torsion element of $G_{2,1}$ in semi-normal form with respect to the basis Y. We define the cycle type of ψ to be the sequence of lengths of finite orbits on Y under the action of ψ.*

Definition 33. *Let Y be a basis for $V_{2,1}$. We define $e_d(Y)$ to be the uniform d-fold expansion of Y.*

Proposition 34. *Let ψ and φ be torsion elements of $G_{2,1}$ in semi-normal form with respect to the bases X and Y. Then, ψ is conjugate to φ if and only if ψ and φ have the same cycle type.*

Proof. If ψ and φ have the same cycle type for the ψ-orbits and φ-orbits then we can construct a map that extends to an element of $G_{2,1}$ conjugating ψ to φ, see Lemma 82 for more details. That is, the number of orbits of length i is either zero or non-zero for both ψ and φ.

If ψ is conjugate to φ, we can construct a map from either $X \to e_d(Y)$ (if $|Y| < |X|$) or $e_d(X) \to Y$ (if $|X| < |Y|$). Then, as our map needs to preserve the orbits on X and on Y, if there is an orbit of size n on X under the action of ψ there needs to be at least one orbit of size n on the basis Y under the action of φ. Thus, ψ and φ must have the same cycle type. □

We now look at regular infinite elements of the group $G_{2,1}$ *i.e.* $V_{P,\psi} = V_{P,\phi} = \emptyset$.

Definition 35. *Let ψ be a regular infinite element of $G_{2,1}$ in semi-normal form with respect to X. For all elements $x \in X$ of type (B), we find the corresponding $m_x \in \mathbb{Z}$ and $\Gamma_x \in \langle A \rangle$ (characteristic multiplier) such that $x\psi^{m_x} = x\Gamma_x$.*

The set

$$\mathcal{M}_\psi = \{(m_x, \Gamma_x) | x \in X, \text{ is an element of type (B)})\}$$

of pairs is called the set of characteristic multipliers and powers for ψ.

Definition 36. *Let ψ be a regular infinite element in quasi-normal form with respect to X. The equivalence relation on the elements of X, $\equiv$, is defined to be the least equivalence relation such that $x \equiv x'$ whenever some $x\Gamma$ and $x'\Delta$ are in the same orbit of ψ, for $\Gamma, \Delta \in \langle A \rangle$.*

Lemma 37. *Suppose that ψ and φ are conjugate regular infinite elements of $G_{2,1}$. Then the set of characteristic multipliers and powers $\mathcal{M}_\psi$, $\mathcal{M}_\varphi$ for ψ and φ coincide.*

Proof. Let X, Y be bases such that ψ, φ are in semi-normal form with respect to them. For the basis X we can look at the equivalence relation, as in Definition 36, on the elements for this basis.

For each equivalence class $\mathcal{X}_j$ we pick an element x in $\mathcal{X}_j$ of type (B) (see Lemma 22). Then, x is a proper characteristic element by Definition 23 of a proper power m_x of ψ with some multiplier Γ_x *i.e.* $x\psi^{m_x} = x\Gamma_x$. If an isomorphism ρ exists (*i.e.* ψ is conjugated to φ by ρ) then $x\rho$ will be a characteristic element of φ^{m_x} with multiplier Γ_x. Thus, $x\rho$ must, by Lemma 22, belong to a semi-infinite orbit of φ. (If no orbit of appropriate characteristic exists, ρ did not exist.)

Therefore, we can look at each semi-infinite orbit and see whether or not its elements are characteristic elements with the "right" multiplier (if one is, they all by the definition of semi-infinite orbit).

Hence, the set of characteristic multipliers for ψ and φ coincide. □

We now have two lemmas which will be useful for the remainder of the subsection.

Lemma 38. *Suppose ψ is a regular infinite element of $G_{2,1}$ in semi-normal form with respect to X and $X\psi = Z$.*

If $X = \coprod_{i=1}^{n} \mathcal{X}_i$ and $Z = \coprod_{i=1}^{m} \mathcal{Z}_i$, where the $\mathcal{X}_i$ and $\mathcal{Z}_i$ are the equivalence classes defined on X and Z (see Definition 36) under the action of ψ respectively, then $n = m$ and ψ maps the equivalence classes on X to the equivalence classes on Z bijectively.

Proof. By definition of the equivalence relation, for $x_i, x_j \in X$, $x_i \equiv x_j$ if and only if $x_i\Gamma\psi^k = x_j\Delta$ for $\Gamma, \Delta \in \langle A \rangle$. Thus, for $x_i\psi = z_i$ and $x_j\psi = z_j$ we have

$$x_i\Gamma\psi^k\psi = x_i\psi\Gamma\psi^k = z_i\Gamma\psi^k \text{ and } x_j\Delta\psi = x_j\psi\Delta = z_j\Delta.$$

Hence, $z_i \equiv z_j$ and it follows that the number of equivalence classes for X under ψ must equal the number of equivalence classes for Z under ψ, as ψ is a bijection between X and Z.

Furthermore, by the above algebra, ψ must induce a bijection between the equivalence classes defined on X and Z under the action of ψ. □

Lemma 39. *Let ψ be as in Lemma 38. Suppose that $\theta_1, ..., \theta_n$ are maps such that each θ_i is defined to be the map that maps the equivalence class $\mathcal{X}_i$ to the equivalence class $\mathcal{Z}_i$ (induced by ψ) and, for all other $x \in X \setminus \mathcal{X}_i$, $x \mapsto x$, for each $i = 1, ..., n$.*

Then, each θ_i can be extended to an element of $G_{2,1}$ such that θ_i commutes with ψ and for all $j \neq i$, θ_i commutes with θ_j.

Proof. By Lemma 38, ψ maps $\mathcal{X}_i$ to $\mathcal{Z}_i$ for each equivalence class $\mathcal{X}_i$ and $\mathcal{Z}_i$ on X and Z respectively.

The action of $\langle\psi\rangle$ on $X\langle A\rangle$ splits $X\langle A\rangle$ into a disjoint union of subsets $X_i\langle A\rangle$ of $X\langle A\rangle$ *i.e.* $X = \coprod_{i=1}^{n} X_i$.

For each equivalence class $\mathcal{X}_i$ of X, it is clear (by the definition of the equivalence relation $\equiv$) that for all $x_i \in \mathcal{X}_i$, $x_i \in X_i$. Thus, we can define each map θ_i by

$$x\theta_i = \begin{cases} x\psi & \text{if } x \in X_i, \\ x & \text{if } x \in X_j \text{ for } i \neq j. \end{cases}$$

Thus, $X_i\psi = Z_i$ (for some similar splitting of $Z\langle A\rangle$ under the action of $\langle\psi\rangle$) where the elements of Z_i come from the equivalence class $\mathcal{Z}_i$ for each i. Hence, each θ_i is a bijective map from X to $(X \setminus X_i) \cup Z_i$.

If $(X \setminus X_i) \cup Z_i$ is not a basis for each map θ_i, then for some $x_i \in X_i$ and $x_j \notin X_i$ for $j \neq i$, $x_i\psi^k = z_i = x_j\Gamma$. However, this would imply that $x_j \equiv x_i$ and $x_j \in X_i$, a contraction.

Therefore, each θ_i is an element of $G_{2,1}$ and it is clear (by definition of each θ_i's) that $\theta_i\psi = \psi\theta_i$ and for all $j \neq i$, $\theta_i\theta_j = \theta_j\theta_i$. □

Remark 40. We note that, if ψ and φ are conjugate by a conjugator ρ and θ commutes with ψ, then $\theta\rho$ is also a conjugator.

For the next two lemmas we assume that ψ, φ are regular infinite element of $G_{2,1}$, in semi-normal form with respect to X and Y respectively.

Lemma 41. *If ψ, φ are conjugate, then there exists a conjugator ρ such that for each equivalence class $\mathcal{X}_i$ defined on X under the action of ψ (see Definition 36), there exists an element x_i of type (B) in $\mathcal{X}_i$ such that $x_i\rho$ is an element y_i of type (B) in Y for φ.*

Proof. Since ψ and φ are conjugate, by Lemma 37 the set of characteristic multipliers for ψ and φ coincide. Let x_i be chosen element of type (B) in $\mathcal{X}_i$. Let ρ be any conjugator, then for some initial or terminal element y_i in Y (with the same characteristic multiplier as x_i) of a semi-infinite orbit and some $j_i \in \mathbb{Z}$ we have

$$x_i\rho = y_i\varphi^{j_i}.$$

Thus, as ρ is a conjugator, we can rewrite this as,

$$x_i\rho\varphi^{-j_i} = x_i\psi^{-j_i}\rho = y_i.$$

For each equivalence class $\mathcal{X}_i$, we define each θ_i as in Lemma 39 and ρ by

$$\rho' = (\prod_{i=1}^{n}\theta_i^{-j_i})\rho,$$

which is an element of $G_{2,1}$ which conjugates ψ to φ by definition (since $\prod_{i=1}^{n}\theta_i^{-j_i}$ commutes with ψ).

We check, for each chosen $x_i \in \mathcal{X}_i$,

$$x_i\rho' = x_i(\prod_{i=1}^{n}\theta_i^{-j_i})\rho = x_i\theta_i^{-j_i}\rho = x_i\psi^{-j_i}\rho = y_i.$$

□

Lemma 42. *Suppose ψ, φ are conjugate and assume that ρ is as in Lemma 41.*

Then there exists a finite set $\mathcal{Y}'_k$ of elements y' of Y such that for all other type (B) elements x' in each equivalence class $\mathcal{X}$, $x'\rho = y'\varphi^l$ for $l \in \mathbb{Z}$. Furthermore, one can determine l uniquely and hence there are only a finite number of possibilities for the image of each x' under ρ.

Proof. We will look at each equivalence class $\mathcal{X}_i$ separately. Thus, without loss of generality, choose x to be an element of type (B) in $\mathcal{X}$ such that $x\rho = y$, $y \in Y$ and element of type (B).

Suppose that $x\Gamma$ and $x'\Delta$ are in the same orbit (*i.e.* $x, x' \in \mathcal{X}$), then $x'\Delta = x\Gamma\psi^k$ and so x' is also of type (B).

Since, by Proposition 37, the set of characteristic multipliers for ψ and φ coincide and the number of elements of type (B) in X and Y under the action of ψ and φ respectively is finite, if ρ_0 extends to a conjugator ρ then there is a finite set $\mathcal{Y}'_k$ of elements y' for each of which $x'\rho = y'\varphi^l$ for some l.

Given that ρ is an isomorphism and knowing $x\rho$ is an element $y \in \mathcal{Y}_k$ and $y' \in \mathcal{Y}'_k$ such that $x'\rho = y'\varphi^l$ then we can determine l uniquely. If $x'\rho = y'\varphi^l$, $x'\Delta = (x\Gamma)\psi^k$ and $\psi\rho = \rho\varphi$ we have,

$$(y'\Delta)\varphi^l = (x'\Delta)\rho = (x\Gamma)\psi^k\rho = (x\rho)\Gamma\varphi^k.$$

Thus, $y'\Delta$ and $(x\rho)\Gamma$ are in the same orbit of φ, as φ is regular infinite element and $Y\langle A\rangle$ decomposes into a disjoint union of infinite orbits under φ.

Thus, if $y'\Delta$ and $(x\rho)\Gamma$ are in the same orbit there will be a unique l such that

$$y'\Delta\varphi^l = (x\rho)\Gamma\varphi^k,$$

by Lemma 27 (ii). Thus we can find a unique l so that $x'\rho = y'\varphi^l$. Hence (as ρ is an isomorphism and $x\rho$ is a specified element of $\mathcal{Y}_k$) $\mathcal{Y}'_k$ is thus a finite set and thus there are only a finite number of possibilities for $x'\rho$.

We can test whether, in fact, there are any possibilities for $x'\rho$ by using Lemma 27 (ii) to see for each $y' \in \mathcal{Y}'_k$ if $y'\Delta$ and $(x\rho)\Gamma$ are in the same orbit of φ.

Given that ρ is an isomorphism and $x\rho$ is in $\mathcal{Y}_k$ (a finite set), then there are only finitely many x' of type (B) in $\mathcal{X}$ satisfying the above conditions. Thus the number of possibilities for x' under ρ is finite. □

We finish this section constructing a finite set of maps, $\mathcal{R}(\psi;\varphi)$, whose elements can be checked if they extend to an element of the group $G_{2,1}$ which conjugates ψ to φ.

Proposition 43. *Let ψ and φ be regular infinite elements of $G_{2,1}$ in semi-normal form with respect to X and Y.*

If the set of characteristic multipliers for ψ and φ coincide, then there exists a finite set $\mathcal{R}(\psi;\varphi)$ of maps from X into $Y\langle A\rangle\langle\lambda\rangle$ such that if ψ is conjugate to φ then there is a conjugator ρ (which we can consider as an isomorphism from $X\langle A\rangle\langle\lambda\rangle$ to $Y\langle A\rangle\langle\lambda\rangle$) for which the restriction $\rho|_X = \rho_0$ is one of the maps in the finite set $\mathcal{R}(\psi;\varphi)$.

Proof. We shall show how to construct the finite set $\mathcal{R}(\psi;\varphi)$ of maps ρ_0 of X into $Y\langle A\rangle\langle\lambda\rangle$ such that if ψ is conjugated to φ then, for some ρ_0 in $\mathcal{R}(\psi;\varphi)$, there is an isomorphism

$$\rho : X\langle A\rangle\langle\lambda\rangle \to Y\langle A\rangle\langle\lambda\rangle$$

such that

$$\rho|_X = \rho_0 \text{ and } \varphi = \rho^{-1}\psi\rho.$$

We can clearly test for a given ρ_0 if it uniquely extends to an isomorphism between the algebras $X\langle A\rangle\langle\lambda\rangle, Y\langle A\rangle\langle\lambda\rangle$ such that $\varphi = \rho^{-1}\psi\rho$.

Since we have assumed that the set of characteristic multipliers for ψ and φ coincide, by Lemma 41 there exists a conjugator ρ such that for each equivalence class $\mathcal{X}$ defined on X under the action of ψ (see Definition 36), there exists an element $x \in \mathcal{X}$ such that $x\rho$ is an element y of type (B) in Y for φ.

We define ρ_0 to be the map which maps the selected $x \in \mathcal{X}$ to the selected $y \in Y$. There are only finitely many initial choices of $x \in \mathcal{X}$ and the image $x\rho$ in Y as the number of elements of type (B) in X and Y is finite.

Next, we wish to show that if x' is another element of type (B) in $\mathcal{X}_j$ then there are only finitely many possibilities for $x'\rho_0$.

This is accomplished by Lemma 42. Which states that there exists a finite set $\mathcal{Y}'_k$ of elements y' of Y such that for all other type (B) elements $x'_i \in \mathcal{X}_i$, $x'\rho = y'\varphi^l$ for some uniquely determined $l \in \mathbb{Z}$. Hence, there is only a finite number of possibilities of each x'_i under ρ.

Hence, we define ρ_0 to be the map which maps the selected $x' \in \mathcal{X}$ to the selected $x'\rho$. There are only finitely many choices of type (B) elements x' in $\mathcal{X}$ and only finitely many choices for the image $x'\rho$ in $Y\langle A\rangle\langle\lambda\rangle$ as the number of elements of type (B) in X and Y is finite. Thus, there are only finitely many maps ρ_0 which can be defined with the above properties.

Note that in using the transitivity of $\equiv$ to show that $x \equiv x'$ where both x and x' are of type (B), we need not go through an element of type (C). For if z is of type (C), then by Lemma 22 there is a $z_1\Gamma_{z_1}$ in the orbit of z for some z_1 of type (B), and any orbit containing some $z\Delta_z$ contains also $z_1\Gamma_{z_1}\Delta_z$. Finally, if $z \equiv x$ and z is of type (C) then $z = z_1\Gamma_{z_1}\psi^k$ for some z_1 of type (B), Γ_{z_1} and k, with $z\rho$ determined once $(z_1\rho)\Gamma_{z_1}\psi^k$ is chosen.

Since X is finite, there are a finite number of equivalence classes. Repeating the above argument for each j, we see that we have constructed a finite set $\mathcal{R}(\psi;\varphi)$ of maps, ρ_0 between X and $Y\langle A\rangle\langle\lambda\rangle$. □

We thus have the following consequence of the above propositions.

Corollary 44. *[9, part of Theorem 9.3] The conjugacy problem is soluble in $G_{2,1}$.*

5.2. *The simultaneous conjugacy problem for regular infinite elements*

We can now extend part of the solution of the conjugacy problem to the simultaneous conjugacy problem for the group $G_{2,1}$.

A group G has soluble simultaneous conjugacy problem if there exists an algorithm such that given any two k-tuples of elements in G,

$$\mathbf{y} = (y_1, ..., y_k) \text{ and } \mathbf{z} = (z_1, ..., z_k),$$

we can determine whether there exists $g \in G$ such that,

$$g^{-1}y_i g = z_i \text{ for } i = 1, ..., k.$$

We write $g^{-1}\mathbf{y}g = \mathbf{z}$ and say that the k-tuple $\mathbf{y}$ is simultaneously conjugate to the k-tuple $\mathbf{z}$.

Definition 45. *A k-tuple of regular infinite elements in $G_{2,1}$, $\boldsymbol{\psi} = (\psi_1, ..., \psi_k)$, is said to be in tuple-quasi-normal form with respect to the basis X if each element ψ_i is in semi-normal form with respect to X but some elements ψ_i would not be in semi-normal form with respect to any proper contraction of X.*

The solution of the simultaneous conjugacy problem for regular infinite elements is similar to the single conjugacy problem for regular infinite elements. However, we first need to construct a "special" basis for the second k-tuple $\boldsymbol{\varphi} = (\varphi_1, ..., \varphi_k)$.

The following lemmas (46 to 53) outline the construction of the "special" basis, which is then defined in Proposition 55.

The first lemmas (and corollaries) are about solving special types of equations in the free algebra $V_{2,1}$.

Lemma 46. *Let φ_1 and φ_2 be regular infinite elements of $G_{2,1}$ in semi-normal form with respect to the basis Y. Let y_1, y_2 be elements of Y which are either initial or terminal elements in semi-infinite orbits with respect to φ_1 and φ_2 respectively and $\Delta_1, \Delta_2 \in \langle A \rangle$.*

If the orbits $y_1\Delta_1\langle\varphi_1\rangle$ and $y_2\Delta_2\langle\varphi_2\rangle$ intersect, then we can determine precisely where they intersect in $\mathbf{x}\langle A \rangle$.

Proof. Given that y_1 and y_2 are either initial or terminal elements in semi-infinite orbits with respect to φ_1 and φ_2 they are type (B) elements in Y for φ_1 and φ_2 respectively. Therefore, we have $y_1\varphi_1^{m_1} = y_1\Gamma_1$ and $y_2\varphi_2^{m_2} = y_2\Gamma_2$ for $\Gamma_1, \Gamma_2 \in \langle A \rangle$. If $y_1\Delta\varphi_1^{a_1} = y_2\Delta\varphi_2^{a_2}$ we would like to be able to determine possibilities for a_1 and a_2.

We first look at the orbit of y_1 and y_2 under φ_1 and φ_2 respectively. We may assume (without loss of generality) that we are looking at two right semi-infinite orbits,

$$..., y_1\varphi_1^{-1}, y_1, y_1\varphi_1, y_1\varphi_1^2, ..., y_1\varphi_1^{m_1} = y_1\Gamma_1, y_1\Gamma_1\varphi_1, y_1\Gamma_1\varphi_1^2, ... \tag{18}$$

and

$$..., y_2\varphi_2^{-1}, y_2, y_2\varphi_2, y_2\varphi_2^2, ..., y_2\varphi_2^{m_1} = y_2\Gamma_2, y_2\Gamma_2\varphi_2, y_2\Gamma_2\varphi_2^2, ... \tag{19}$$

We can rename part of Sequence (18),

$$..., y_{1,-1}, y_1 = y_{1,0}, y_{1,1}, y_{1,2}, ..., y_{1,m_1-1}, y_{1,0}\Gamma_1, y_{1,1}\Gamma_1, y_{1,2}\Gamma_1, ... \tag{20}$$

and Sequence (19),

$$..., y_{2,-1}, y_2 = y_{2,0}, y_{2,1}, y_{2,2}, ..., y_{2,m_2-1}, y_{2,0}\Gamma_2, y_{2,1}\Gamma_2, y_{2,2}\Gamma_2, ... \tag{21}$$

Since y_1 and y_2 are initial elements in a semi-infinite orbit in $Y\langle A \rangle$ under the action of φ_1 and φ_2 respectively, we see that $y_{1,0}, ..., y_{1,m_1-1}, y_{2,0}, ..., y_{2,m_2-1} \in Y$. Therefore, if $y_1\varphi_1^{a_1} = y_2\varphi_2^{a_2}$ (Δ_1, Δ_2 are trivial) then, firstly, $\Gamma_1^{n_1} = \Gamma_2^{n_2}$ for $n_1, n_2 \in \mathbb{N}$, $\gcd(n_1, n_2) = 1$ and, secondly,

$$y_1\varphi_1^{a_1} = y_{1,i}\Gamma_1^{n_1} \text{ and } y_2\varphi_2^{a_2} = y_{2,j}\Gamma_2^{n_2},$$

Since the equation $\Gamma_1^{n_1} = \Gamma_2^{n_2}$ can be solved for $n_1, n_2 \in \mathbb{N}$, $\gcd(n_1, n_2) = 1$ we can determine the precise location in $\mathbf{x}\langle A \rangle$ where the orbits of y_1 and y_2 intersects. If the orbits intersect once, they intersect infinitely many times (as $\Gamma_1^{Kn_1} = \Gamma_2^{Kn_2}$ for $K \in \mathbb{N}$).

If $y_1\Delta_1\varphi_1^{a_1} = y_2\Delta_2\varphi_2^{a_2}$ (for non-trivial Δ_1, Δ_2) then, since φ_1, φ_2 are automorphisms, $y_1\varphi_1^{a_1}\Delta_1 = y_2\varphi_2^{a_2}\Delta_2$. This implies, firstly, that $\Gamma_1^{n_1}\Delta_1 = \Gamma_2^{n_2}\Delta_2$ for $n_1, n_2 \in \mathbb{N}$ and, secondly,

$$y_1\varphi_1^{a_1} = y_{1,i}\Gamma_1^{n_1} \text{ and } y_2\varphi_2^{a_2} = y_{2,j}\Gamma_2^{n_2},$$

where $a_1 = i + m_1n_1$ and $a_2 = j + m_2n_2$, implies $y_{1,i} = y_{2,j}$.

Since the equation $\Gamma_1^{n_1}\Delta_1 = \Gamma_2^{n_2}\Delta_2$ can be solved for $n_1, n_2 \in \mathbb{N}$, we can determine the precise location in $\mathbf{x}\langle A\rangle$ where the orbits $y_1\Delta_1\langle\varphi_1\rangle$ and $y_2\Delta_2\langle\varphi_2\rangle$ intersect.

Let y_3 be a terminal element of a left semi-infinite orbit under φ_2 respectively, then

$$..., y_3\varphi_2, y_3, y_3\varphi_2^{-1}, y_3\varphi_2^{-2}, ..., y_3\varphi_2^{-m_1} = y_3\Gamma_3, y_3\Gamma_3\varphi_2^{-1}, y_3\Gamma_3\varphi_2^{-2}, ... \tag{24}$$

We can rename part of Sequence (24),

$$..., y_{3,-1}, y_3 = y_{3,0}, y_{3,1}, y_{3,2}, ..., y_{3,m_1-1}, y_{3,0}\Gamma_3, y_{3,1}\Gamma_3, y_{3,2}\Gamma_3, ... \tag{25}$$

We can, therefore, follow the same procedure as above to determine the precise intersection of the orbits $y_1\Delta_1\langle\varphi_1\rangle$ and $y_3\Delta_2\langle\varphi_2\rangle$. □

We can form a corollary to the above lemma. Let $\Delta_i \in \langle A\rangle$ (possibly trivial) for $i = 1, ..., k$ and $y_1, .., y_k$ elements of Y which are either initial or terminal elements in semi-infinite orbits for $\varphi_1, ..., \varphi_k$.

Corollary 47. *If the orbits $y_1\Delta_1\langle\varphi_1\rangle$, ..., $y_k\Delta_k\langle\varphi_k\rangle$ intersect then we can determine precisely where they intersect in $\mathbf{x}\langle A\rangle$.*

By the above result, it is natural to give a name to the points in $\mathbf{x}\langle A\rangle$ where the orbits $y_1\Delta_1\langle\varphi_1\rangle$, ..., $y_k\Delta_k\langle\varphi_k\rangle$ (for possibly trivial $\Delta_i \in \langle A\rangle$) intersect.

Definition 48. *Let $y_1, .., y_k$ be elements of Y which are either initial or terminal elements in semi-infinite orbits for $\varphi_1, ..., \varphi_k$. If the orbits $y_1\Delta_1\langle\varphi_1\rangle$, ..., $y_k\Delta_k\langle\varphi_k\rangle$ intersect, then we call this set the intersection set and the elements of this set* intersection points.

For the next two lemmas, we assume that the k-tuples $\boldsymbol{\psi} = (\psi_1, ..., \psi_k)$ and $\boldsymbol{\varphi} = (\varphi_1, ..., \varphi_k)$ of regular infinite elements of $G_{2,1}$ are pairwise conjugate (*i.e.* ψ_i is conjugate to φ_i for all i) and in tuple-quasi-normal form with respect to X and Y respectively.

Let $\mathcal{X}_{j,i}$ be the equivalence classes of elements of X defined by the action of ψ_i for $i = 1, ..., k$.

Lemma 49. *Let $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ be simultaneously conjugate by some conjugator ρ. For fixed i and j, suppose that $x_{j,i}$ in $\mathcal{X}_{j,i}$ is an element of type (B) for ψ_i and $y_{j,i}$ in Y is an initial or terminal element in a semi-infinite orbit under the action of φ_i such that $x_{j,i}\rho$ is in the orbit $y_{j,i}\langle\varphi_i\rangle$.*

Then, for each (ψ_s, φ_s) pair ($s \neq i$),

(1) either $x_{j,i}$ is an element of type (B) for ψ_s and there exists y' in Y an initial or terminal element in a semi-infinite orbit under the action of φ_s such that the intersection set $\mathcal{I}(i, j, s) = y_{j,i}\langle\varphi_i\rangle \cap y'\langle\varphi_s\rangle \neq \emptyset$;

(2) or $x_{j,i}$ is an element of type (C) for ψ_s and there exists $\Gamma \in \langle A\rangle$ and y' in Y an initial or terminal element in a semi-infinite orbit under the action of φ_s such that the intersection set $\mathcal{I}(i, j, s) = y_{j,i}\langle\varphi_i\rangle \cap y'\Gamma\langle\varphi_s\rangle \neq \emptyset$.

Proof. Since $x_{j,i}\rho = y_{j,i}\varphi_i^{k_{j,i}}$, and ρ is a simultaneous conjugator, for $s \neq i$,

(1) if $x_{j,i}$ is an element of type (B) for ψ_s, we know that for some initial or terminal element y' in Y of a semi-infinite orbit for φ_s,

$$x_{j,i}\rho = y'\varphi_s^k.$$

Thus, as ρ is a simultaneous conjugator,

$$y'\varphi_s^k = y_{j,i}\varphi_i^{k_{j,i}},$$

for $y_{j,i}$ an initial or terminal element in Y of a semi-infinite orbit for φ_i. Thus, the orbits of $y'\langle\varphi_s\rangle$ and $y_{j,i}\langle\varphi_i\rangle$ intersect.

(2) if $x_{j,i}$ is an element of type (C) for ψ_s, then firstly,

$$x_{j,i}\psi_s^n = x'\Gamma$$

for $\Gamma \in \langle A\rangle$, minimal $n \in \mathbb{Z}$ and x' some element of type (B) in X under the action of ψ_s. We know that for some initial or terminal element y' in Y of a semi-infinite orbit for φ_s,

$$x'\rho = y'\varphi_s^k.$$

As,

$$x'\Gamma\rho = x'\rho\Gamma = y'\varphi_s^k\Gamma,$$

we have

$$x_{j,i}\psi_s^n\rho = y'\Gamma\varphi_s^k$$

and (given that ρ is a simultaneous conjugator),

$$x_{j,i}\rho = y'\Gamma\varphi_s^k\varphi_s^{-n} = y'\Gamma\varphi_s^{k-n}$$

Thus, as ρ is a simultaneous conjugator,

$$y'\Gamma\varphi_s^{k-n} = y_{j,i}\varphi_i^k,$$

for $y_{j,i}$ an initial or terminal element in Y of a semi-infinite orbit for φ_i. Thus, the orbits $y'\Gamma\langle\varphi_s\rangle$ and $y_{j,i}\langle\varphi_1\rangle$ intersect. □

Applying Lemma 49 for each $i = 1, ..., k$ and over all equivalence classes $\mathcal{X}_{j,i}$ defined on X under each ψ_i, gives rise to the following definition.

Definition 50. *We define the set,*

$$\mathcal{I}(\boldsymbol{\psi},\boldsymbol{\varphi}) := \bigcap_{i=1}^{k}\bigcup_{j}\bigcup_{s\neq i}\mathcal{I}(i,j,s)$$

of intersection points of the pair of k-tuples $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$.

We can now form the obvious corollary by applying Lemma 49 for each $i = 1, ..., k$ and over all equivalence classes $\mathcal{X}_{j,i}$ defined on X under the action of each ψ_i.

Corollary 51. *If $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ are simultaneously conjugate, then the set $\mathcal{I}(\boldsymbol{\psi}, \boldsymbol{\varphi})$ of intersection points is non-empty.*

We should make some general comments about Corollary 51. Firstly, recall that the basis X which was used for Lemma 49 was the smallest basis giving all elements of the k-tuple $\boldsymbol{\psi}$ in semi-normal form (see Definition 45). This is then, in some sense, the smallest basis that we can define the action of ρ (a simultaneous conjugator) on.

Definition 52. *Let $\mathcal{B}(X)$ be the set of all elements x in X, such that x is of type (B) for some element in $\boldsymbol{\psi}$.*

Secondly, a necessary condition for ρ to be a simultaneous conjugator is that, for every element $x_l \in \mathcal{B}(X)$ we have that,

$$x_l\rho = y_{l,i_1}\varphi_{i_1}^{k_{l,i_1}} = ... = y_{l,i_m}\varphi_{i_m}^{k_{l,i_m}} = y' \in Y\langle A\rangle, \tag{35}$$

where $k_{l,i_1}, .., k_{l,i_m} \in \mathbb{Z}$ and $y_{l,i_1}, ..., y_{l,i_m}$ elements in Y that are initial or terminal elements in a semi-infinite orbit for $\varphi_{i_1}, ..., \varphi_{i_m}$. The set of all elements y' in $Y\langle A\rangle$ which are found by the intersection of the orbits $y_{l,i_1}\langle\varphi_{i_1}\rangle, ..., y_{l,i_m}\langle\varphi_{i_m}\rangle$ will be called the *set of solutions to equation 35 for x_l* and denoted Y_{x_l} (this set could be infinite). It is clear that $Y_{x_l} \subset \mathcal{I}(\boldsymbol{\psi}, \boldsymbol{\varphi})$.

We can thus identify subsets of $\mathcal{I}(\boldsymbol{\psi}, \boldsymbol{\varphi})$ (namely the sets Y_{x_l}) with the set $\mathcal{B}(X)$. Specifically, we can identify each $x_l \in \mathcal{B}(X)$ with a set Y_{x_l}.

Thirdly, recall that the basis Y which was used for Lemma 49 was the smallest basis giving all elements of the k-tuple $\boldsymbol{\varphi}$ in semi-normal form (see Definition 45).

If a simultaneous conjugator ρ exists, we can always extend the action of ρ on the basis X to the action of ρ on an expansion of X, $\widetilde{X}$, such that we get a bijection,

$$\rho : \widetilde{X} \to \widetilde{Y},$$

where $\widetilde{Y}$ is some expansion of Y. This is achieved by the solution to the single conjugator problem, once we have chosen where to send the elements $x \in \mathcal{B}(X)$ in $Y\langle A\rangle$.

We will explain how to form the expansion $\widetilde{X}$ later. For now, we state some conditions for the construction of a basis $\widetilde{Y}$. That is, if the construction of $\widetilde{Y}$ fails, then the k-tuples $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ are not simultaneously conjugate.

Lemma 53. *Suppose for each $x \in \mathcal{B}(X)$ one chooses some element $x\rho$ that is a solution to equation (35)* i.e. *$x\rho \in Y_x$. Then, the set $\{x\rho | x \in \mathcal{B}(X)\}$ can be completed to a basis $\widetilde{Y}$ in $Y\langle A\rangle$.*

Proof. We have chosen, for each $x \in \mathcal{B}(X)$, some element $x\rho \in Y_x$. Then, the only obstruction to completing this set $\{x\rho | x \in \mathcal{B}(X)\}$ of elements (in $Y\langle A\rangle$, as $x\rho \in \mathcal{I}(\boldsymbol{\psi}, \boldsymbol{\varphi})$) to a basis is if there exists $\Gamma \in \langle A\rangle$ such that for $x_1, x_2 \in \mathcal{B}(X)$,

$$x_1\rho = x_2\rho\Gamma.$$

However, since ρ (if it exists) is an automorphism of the free algebra $V_{2,1}$,

$$x_2\rho\Gamma = x_2\Gamma\rho.$$

Therefore, $x_1 = x_2\Gamma$. This is a contraction, as the elements of $\mathcal{B}(X)$ form are part of the basis X. □

By the definition of left and right semi-infinite orbits (see Section 3.1) of elements of $\widetilde{Y}$, under the action of $\varphi_{i_1}, ..., \varphi_{i_m}$, we have that the elements y' defined above are initial or terminal elements of a semi-infinite orbit under the action of $\varphi_{i_1}, ..., \varphi_{i_m}$, with respect to the subalgebra $\widetilde{Y}\langle A\rangle$. (There are, obviously, other elements in $\widetilde{Y}$ that are not initial or terminal elements in a semi-infinite orbit, they will be elements of type (C) for all φ_i.)

We will now form a definition about a special expansion of the basis Y, on our way to a proof of the simultaneous conjugacy problem for regular infinite elements of $G_{2,1}$.

Definition 54. *Define $\widetilde{Y}_0$ to be the smallest expansion of Y which consists of elements y' of $\mathcal{I}(\boldsymbol{\psi}, \boldsymbol{\varphi})$ that are also initial or terminal elements in semi-infinite orbits for some element of $\boldsymbol{\varphi}$, with respect to the subalgebra $\widetilde{Y}_0\langle A\rangle$.*

For the next two results, we will assume that the k-tuples $\boldsymbol{\psi} = (\psi_1, ..., \psi_k)$ and $\boldsymbol{\varphi} = (\varphi_1, ..., \varphi_k)$ of regular infinite elements of $G_{2,1}$ are pairwise conjugate (*i.e.* ψ_i is conjugate to φ_i for all i) and in tuple-quasi-normal form with respect to X and Y respectively.

Also, let $\mathcal{X}_{j,i}$ be the equivalence classes of elements of X defined by the action of ψ_i for $i = 1, ..., k$.

Proposition 55. *Suppose $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ are simultaneously conjugate. Then, there exits a conjugator ρ and a computable minimal expansion $\widetilde{Y}$ of Y such that, for each chosen type (B) element $x_{j,i}$ in each equivalence class $\mathcal{X}_{j,i}$ for each ψ_i,*

$$x_{j,i}\rho = y_{j,i}$$

for $y_{j,i}$ an initial or terminal element in $\widetilde{Y}$ in a semi-infinite orbit for some φ_i.

Furthermore, the minimal expansion $\widetilde{Y}$ of Y with this property is $\widetilde{Y}_0$ from Definition 54.

Proof. Given that $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ are simultaneous conjugate, for all $x_l \in \mathcal{B}(X)$, $x_l\rho$ must be a solution to equation (35). By Lemma 53, for all $x_{j,i} \in \mathcal{B}(X)$ such a set of solutions $\{x\rho | x \in \mathcal{B}(X)\}$ can be completed to a basis $\widetilde{Y}$ (an expansion of Y) such that

$$x_{j,i}\rho = y_{j,i},$$

where $y_{j,i}$ in $\widetilde{Y}$ is an initial or terminal element in a semi-infinite orbit under the action of a least one φ_i, with respect to the subalgebra $\widetilde{Y}\langle A\rangle$.

We need to show that such a basis is constructible and if $\widetilde{Y}_0$ can not be constructed, $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ are not simultaneously conjugate.

Suppose we choose $\widetilde{Y}$ to be the basis $\widetilde{Y}_0$, defined in Definition 54. That is, we define the elements $x_l\rho$ for all $x_l \in \mathcal{B}(X)$ to be the "smallest" element in each Y_{x_l}, call it y'_l. The elements y'_l are solutions to equations (35) and imply that

$$x_l\psi_i\rho = x_l\rho\varphi,$$

for all i.

Let $x_l \in \mathcal{B}(X)$ and $a \in \mathbb{Z}$. If we look at

$$x_l\psi_i^a\rho = x_l\rho\varphi_i^a,$$

then as x_l is of type (B), it will be a characteristic element for $\psi_{i_1}^{n_{i_1}}, \ldots, \psi_{i_m}^{n_{i_m}}$ ($n_{i_p} \in \mathbb{Z}$) with multiplier $\Gamma_{i_p} \in \langle A\rangle$ and $y_l \in \widetilde{Y}_0$ will be a characteristic element for $\varphi_{i_1}^{n_{i_1}}, \ldots, \varphi_{i_m}^{n_{i_m}}$ ($n_{i_p} \in \mathbb{Z}$) with multiplier Γ_{i_p}. Therefore, for each $x_l \in \mathcal{B}(X)$ the equation

$$x_l\psi_i^a\rho = x_l\rho\varphi_i^a,$$

becomes,

$$x_l\Gamma_{i_p}^{t_{i_p}}\psi^{\bar{a}_i}\rho = x_l\rho\varphi_i^a = y_l\varphi_i^a = y_l\Gamma_{i_p}^{t_{i_p}}\varphi_i^{\bar{a}_i}\ (t_{i_p} \in \mathbb{N}),$$

where $\bar{a}_i = a - t_{i_p}n_{i_p}$ with $0 \leq \bar{a}_i \leq n_{i_p} - 1$. Since ψ_i, φ_i and ρ are automorphisms,

$$x_l\Gamma_{i_p}^{t_{i_p}}\psi_i^{\bar{a}_i}\rho = x_l\psi_i^{\bar{a}_i}\rho\Gamma_{i_p}^{t_{i_p}} \text{ and } y_l\Gamma_{i_p}^{t_{i_p}}\varphi_i^{\bar{a}_i} = y_l\varphi_i^{\bar{a}_i}\Gamma_{i_p}^{t_{i_p}}.$$

Therefore,

$$x_l\psi_i^{\bar{a}_i}\rho = y_l\varphi_i^{\bar{a}_i},$$

and there are only finitely many elements $x_l\psi_i^{\bar{a}_i}$ and $y_l\varphi_i^{\bar{a}_i}$, as $0 \leq \bar{a}_i \leq n_{i_p} - 1$. Furthermore, each $x_l\psi_i^{\bar{a}_i} \in \mathcal{B}(X)$ (otherwise, since x_l is of type (B), we would have $x_l\psi_i^{\bar{a}_i} = x_l\psi_i^{b_i}\Delta_i$). We have already assigned the images of all $x_l \in \mathcal{B}(X)$, such that $x_l\rho$ is a solution to equation (35). Hence,

$$x_l\psi_i^a\rho = x_l\rho\varphi_i^a \text{ for all } a \in \mathbb{Z}.$$

If, after the assignment of each $x_l \in \mathcal{B}(X)$ to the "smallest" element in Y_{x_l}, ρ fails to be a conjugator, then there must exist a type (C) $x'_{j,i}$ element in X, for all ψ_i, such that

$$x'_{j,i}\psi_i\rho \neq x'_{j,i}\rho\varphi_i \text{ for some } i.$$

By definition of type (C) elements (see Lemma 22), there exists an element $x_{j,i}$ of type (B), $\Gamma_{j,i}$ and $k_{j,i} \in \mathbb{Z}$ such that,

$$x'_{j,i}\psi_i^{k_{j,i}} = x_{j,i}\Gamma_{j,i},$$

for all i, j. However, these type (B) elements $x_{j,i}$ satisfy equations of the form (since $x_{i,j}\rho = y'_{j,i} \in Y_{x_{j,i}}$),

$$x_{j,i}\psi_i\rho = x_{j,i}\rho\varphi_i.$$

Thus, for the equations,

$$x'_{j,i}\psi_i\rho = x_{j,i}\Gamma_{j,i}\psi_i^{-k_{j,i}}\psi_i\rho = x_{j,i}\psi_i^{-k_{j,i}}\psi_i\rho\Gamma_{j,i},$$

$$x'_{j,i}\rho\varphi_i = x_{j,i}\Gamma_{j,i}\psi_i^{-k_{j,i}}\rho\varphi_i = x_{j,i}\psi_i^{-k_{j,i}}\rho\varphi_i\Gamma_{j,i},$$

we have that,

$$x_{j,i}\psi_i^{-k_{j,i}}\psi_i\rho = x_{j,i}\rho\varphi_i^{-k_{j,i}}\varphi_i = y'_{j,i}\varphi_i^{-k_{j,i}+1}$$

and

$$x_{j,i}\psi_i^{-k_{j,i}}\rho\varphi_i = x_{j,i}\rho\varphi_i^{-k_{j,i}}\varphi_i = y'_{j,i}\varphi_i^{-k_{j,i}+1}$$

for $y'_{j,i}$ an initial or terminal element in $\widetilde{Y}_0$ in a semi-infinite orbit for φ_i. A contradiction, since $x'_{j,i}\psi_i\rho \neq x'_{j,i}\rho\varphi_i$.

Hence, once we define the basis $\widetilde{Y}_0$, ρ must be a simultaneous conjugator and we can map each chosen type (B) element $x_{j,i}$ in each equivalence class $\mathcal{X}_{j,i}$ for each ψ_i,

$$x_{j,i}\rho = y_{j,i}$$

for $y_{j,i}$ an initial or terminal element in $\widetilde{Y}_0$ in a semi-infinite orbit for some φ_i.

Furthermore, it is clear that the basis $\widetilde{Y}_0$ from Definition 54 is computable. (The remaining elements of $\widetilde{Y}_0$ which are type (C) for all φ_i can filled in by applying the conjugacy problem solution to each (ψ_i, φ_i) pair, now that we have an initial map for the type (B) elements $\mathcal{B}(X)$.) □

We can now follow the same process as for the single conjugacy problem with our newly constructed basis $\widetilde{Y}_0$.

Form now on we assume that the k-tuples $\boldsymbol{\psi} = (\psi_1, ..., \psi_k)$ and $\boldsymbol{\varphi} = (\varphi_1, ..., \varphi_k)$ of regular infinite elements of $G_{2,1}$ are pairwise conjugate (*i.e.* ψ_i is conjugate to φ_i for all i) and $\boldsymbol{\psi} = (\psi_1, ..., \psi_k)$ is in tuple-quasi-normal form with respect to X. We assume that Y is defined as the basis from Proposition 55 giving all elements in the k-tuple $\boldsymbol{\varphi} = (\varphi_1, ..., \varphi_k)$ in semi-normal form.

Lemma 56. *Suppose $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ are simultaneously conjugate and assume that ρ is as in Proposition 55.*

Then there exists a finite set $\mathcal{Y}'_k$ of elements y' of Y for each (ψ_i, φ_i) pair such that for all other type (B) elements x' in each equivalence class $\mathcal{X}$ under ψ_i, $x'\rho = y'\varphi_i^l$ for $l \in \mathbb{Z}$. Furthermore, one can determine l uniquely and hence there are only a finite number of possibilities for the image of each x' under ρ for each (ψ_i, φ_i) pair and subsequently for the tuples $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$.

Proof. We look at each (ψ_i, φ_i) pair separately and apply Lemma 42 to the bases X and Y with ρ mapping an element of type (B) for each equivalence class $\mathcal{X}$ for each (ψ_i, φ_i) pair. This, then, creates a finite set of elements y' of Y for each (ψ_i, φ_i) pair such that for all other type (B) elements x' in each equivalence class $\mathcal{X}$ under ψ_i, $x'\rho = y'\varphi_i^l$ for some uniquely determined $l \in \mathbb{Z}$.

One then checks that such a conjugator is consistent for all other (ψ_j, φ_j) pairs, $i \neq j$. □

Proposition 57. *Let $\boldsymbol{\psi} = (\psi_1, ..., \psi_k)$ and $\boldsymbol{\varphi} = (\varphi_1, ..., \varphi_k)$ be two k-tuples of regular infinite elements of $G_{2,1}$ in tuple-quasi-normal form with respect to X and Y (see Definition 45) respectively.*

Assume that for each i, ψ_i is conjugate to φ_i. If the k-tuple $\boldsymbol{\psi}$ is simultaneously conjugate to the k-tuple $\boldsymbol{\varphi}$ then there exists a finite set $\mathcal{R}(\boldsymbol{\psi};\boldsymbol{\varphi})$ of maps such that there exists a conjugator ρ (which can be seen as an isomorphism from $X\langle A\rangle\langle\lambda\rangle$ to $Y\langle A\rangle\langle\lambda\rangle$) for which the restriction $\rho|_X = \rho_0$ is one of the maps in the finite set $\mathcal{R}(\boldsymbol{\psi};\boldsymbol{\varphi})$.

Proof. We can clearly test for a given ρ_0 if it uniquely extends to an isomorphism between the algebras $X\langle A\rangle\langle\lambda\rangle$, $Y\langle A\rangle\langle\lambda\rangle$ such that $\rho^{-1}\psi_i\rho = \varphi_i$ for all i.

Since we have assumed that the set of characteristic multipliers for ψ_i and φ_i coincide, by Proposition 55 there exists a conjugator ρ such that for each equivalence class $\mathcal{X}$ defined on X under the action of ψ_i (see Definition 36), there exists an element $x_i \in \mathcal{X}$ such that $x_i\rho$ is an element y_i of type (B) in Y for φ_i.

We define ρ_0 to be the map which maps the selected $x \in \mathcal{X}$ to the selected $y \in Y$. There are only finitely many initial choices of $x \in \mathcal{X}$ under each ψ_i and finitely many choices for the image $x\rho$ in Y (as the number of elements of type (B) in X and Y is finite for each pair (ψ_i, φ_i)).

Next, we wish to show that if x' is another element of type (B) in $\mathcal{X}$ for each pair (ψ_i, φ_i) then there are only finitely many possibilities for $x'\rho_0$.

This is accomplished by Proposition 56. Which states that there exists a finite set $\mathcal{Y}'_k$ of elements y' of Y for each (ψ_i, φ_i) pair such that for all other type (B) elements x' in each equivalence class $\mathcal{X}$ under ψ_i, $x'\rho = y'\varphi_i^l$ for some unique $l \in \mathbb{Z}$.

Hence, we define ρ_0 to be the map which maps the selected $x' \in \mathcal{X}$ for each ψ_i to the selected $x'\rho$. There are only finitely many choices of type (B) elements x' in $\mathcal{X}$ for each ψ_i and only finitely many choices for the image $x'\rho$ in $Y\langle A\rangle\langle\lambda\rangle$ for each ψ_i as the number of characteristic elements in X and Y is finite for each pair (ψ_i, φ_i). Thus, there are only finitely many maps ρ_0 which can be defined with the above properties.

Note that in using the transitivity of $\equiv$ to show that $x \equiv x'$ where both x and x' are of type (B), we need not go through an element of type (C). For if z is of type (C), then by Lemma 22 there is a $z_1\Gamma_{z_1}$ in the orbit of z for some z_1 of type (B), and any orbit containing some $z\Delta_z$ contains also $z_1\Gamma_{z_1}\Delta_z$. Finally, if $z \equiv x$ and z is of type (C) then $z = z_1\Gamma_{z_1}\psi_i^k$ for some z_1 of type (B), Γ_{z_1} and k, with $z\rho$ determined once $(z_1\rho)\Gamma_{z_1}\psi_i^k$ is chosen. We can do this for each (ψ_i, φ_i) pair and check the consistency of ρ by looking at all other pairs (ψ_j, φ_j), $i \neq j$.

By the above process, we have constructed a finite set $\mathcal{R}(\boldsymbol{\psi};\boldsymbol{\varphi})$ of maps ρ_0 between X and $Y\langle A\rangle\langle\lambda\rangle$. □

We now have the following consequence to the above proposition.

Theorem 58. *The simultaneous conjugacy problem for regular infinite elements of $G_{2,1}$ is solvable.*

5.3. *Power conjugacy problem*

The power conjugacy problem naturally arises when you have any group B and G an HNN group extension given by

$$G = \langle a, B | \text{rel } B, a^{-1}Wa = V\rangle,$$

where W and V are words in the generators of B defining elements of the same order. It follows from [16, Lemma 5] that if x and y are elements in B that are conjugate in G but not in B then x and y are conjugate in B to powers of W or V and hence in G to powers of W.

This motivates the study of the power conjugate problem for a group.

Problem 59. [1] Does there exists $a, b \in \mathbb{Z}$, $z \in G$ such that $x^a = z^{-1}y^bz \neq 1$ for $x, y \in G$?

It is clear that we should concentrate on the case $|a| \neq |b|$. See [1,13,17] for references to this problem.

The aim of this subsection is to answer this question for the Higman-Thompson group $G_{2,1}$.

We will now prove that the Higman-Thompson group has solvable power conjugacy problem (Theorem 60). We can break this problem down into two cases by Theorem 28: when ψ, φ are torsion elements and when ψ, φ are regular infinite elements.

5.3.1. *Torsion elements*

Let X, Y be bases such that torsion elements ψ, φ are in quasi-normal form with respect to them.

As ψ and φ are of finite order, there are only a finite number of choices for the pair $(a, b) \in \mathbb{Z}^2$ such that $|a| \neq |b|$. We can therefore take all pairs and apply the conjugacy problem criterion for torsion elements, that we use Proposition 34.

5.3.2. *Regular infinite elements*

Let X, Y be bases such that the regular infinite elements ψ, φ are in quasi-normal form with respect to them. If we want to find integers a, b with $|a| \neq |b|$ such that ψ^a is conjugate to φ^b, we need to examine all elements of X (resp. Y) of type (B).

Each element $x \in X$ of type (B) is a proper characteristic element of a proper power m_x of ψ with some multiplier Γ_x *i.e.* $x\psi^{m_x} = x\Gamma_x$. Similarly, for the elements $y \in Y$ of type (B) are proper characteristic element of a proper power m_y of ψ with some multiplier Γ_x *i.e.* $y\varphi^{m_y} = x\Delta_y$. Both m_x and m_y are taken to be the smallest possible integer.

A condition for conjugacy is that the set of characteristic multipliers and powers $\mathcal{M}_\psi$ and $\mathcal{M}_\varphi$ coincide, by Proposition 37. Therefore, as $x\psi^{am_x} = x(\Gamma_x)^a$ and $y\varphi^{bm_y} = x(\Gamma_y)^b$, we need to be able to pair up (m_x, Γ_x) and (m_y, Δ_y) such that

$$a_i m_x = b_j m_y \text{ and } \Gamma_x^{a_i} = \Delta_x^{b_j},$$

for $a_i, b_j \in \mathbb{Z} \setminus \{0\}$. There is only a finite number of possibilities for a_i, b_j if we require that $\gcd(a_i, b_j) = 1$, as the sets $\mathcal{M}_\psi$ and $\mathcal{M}_\varphi$ are finite. We can therefore take all a_i which are coprime a_i' and all b_j which are coprime b_j' and form $a = \prod_i a_i'$ and $b = \prod_j b_j'$.

There will be a finite set of pairs of elements (a, b) satisfying the about condition. We then apply the conjugacy problem solution to ψ^a and φ^b. Thus we get the following.

Theorem 60. *The power conjugacy problem for the Higman-Thompson group $G_{2,1}$ is solvable.*

6. Centralizers of Elements in $G_{2,1}$

Using Theorem 44 we can split the construction of centralizing elements in to elements that centralize regular infinite and periodic elements of $G_{2,1}$.

This section develops the correspondence between finitely generated free-abelian subgroups of $G_{2,1}$ and the centralizing elements for a regular infinite element. The final section provides criteria for an element of $G_{2,1}$ to centralize a given torsion element.

6.1. *Finitely generated free-abelian subgroups of $G_{2,1}$*

This section provides information that will be used in Section 6.2, connecting the centralizer of a regular infinite element of $G_{2,1}$ with the normalizer of finitely generated free abelian subgroups of $G_{2,1}$.

Definition 61. *Let H be a subgroup of $G_{2,1}$. We define the centralizer of H to be*

$$C_{G_{2,1}}(H) = \{\varphi \in G_{2,1} | \forall \psi \in H, \psi\varphi = \varphi\psi\}$$

and the normalizer of H to be

$$N_{G_{2,1}}(H) = \{\varphi \in G_{2,1} | \varphi^{-1} H \varphi = H\}.$$

The next theorem is [9, Theorem 9.9 part (ii)] and the proofs for the subsequent Lemma's follow the proof of [9, Theorem 9.9 part (ii)]. However, it is universally accepted that this proof is difficult to read. Furthermore, more details are required to extend this proof and the tools involved for the scope of this article. Hence, we include a very detailed account of the proof of following part of [9, Theorem9.9].

Theorem 62. *[9, Theorem 9.9 part (ii)] Let H be a finitely generated free-abelian subgroup contained in $G_{2,1}$. Then the centralizer $C_{G_{2,1}}(H)$ is of finite index in the normalizer $N_{G_{2,1}}(H)$.*

The rest of this section is devoted to the proof of the above theorem. From now on we assume that H satisfies the conditions of Theorem 62.

Definition 63. *[9] Let $\psi \in H$. A* characteristic sequence *for ψ is an infinite sequence*

$$\widetilde{u} = ..., u_{-2}, u_{-1}, u_0, u_1, u_2, ... ,$$

of elements of $V_{2,1}$, such that $u_{i+1} = u_i\psi$ and there exist some non-trivial $\Gamma \in \langle A \rangle$ such that

(1) for all i, $u_{i+1} = u_i\Gamma$,
(2) or for all i, $u_i = u_{i+1}\Gamma$.

In the first case we say that $\widetilde{u}$ is a right characteristic sequence and in the second case a left characteristic sequence for ψ.

Remark 64. Notice that if $\widetilde{u}$ is a right characteristic sequence then there exists a smallest i for which $u_i \in \mathbf{x}\langle A \rangle$ and then $u_j \in \mathbf{x}\langle A \rangle$,for all $j \geq i$. In that case u_i is a right characteristic element by Definition 23 and is in a right semi-infinite orbit with respect to $\mathbf{x}$.

Similarly, if $\widetilde{u}$ is in a left characteristic sequence then there exists a biggest i for which $u_i \in \mathbf{x}\langle A \rangle$ and then $u_j \in \mathbf{x}\langle A \rangle$ for all $j \leq i$. In that case u_i is left characteristic element by Definition 23 is in a left semi-infinite orbit with respect to $\mathbf{x}$.

It may also be noted that a right characteristic sequence for ψ is a left characteristic sequence for ψ^{-1} and that a left characteristic sequence for ψ is a right characteristic sequence for ψ^{-1}.

We say that $\widetilde{u}$ is a characteristic sequence for H if it is a characteristic sequence for some element ψ in H.

We will see that $N_{G_{2,1}}(H)$ acts on the set of characteristic sequences for H. We will introduce the notion of an oriented characteristic sequence and that some elements of $N_{G_{2,1}}(H)$ preserve the orientation of the sequence and some reverse the orientation of the sequence.

Lemma 65. *[9]Let $\widetilde{u}$ be a characteristic sequence for $\psi \in H$. If $\phi \in N_{G_{2,1}}(H)$, then $\widetilde{u}\phi$ is a characteristic sequence for $\phi^{-1}\psi\phi$.*

Proof. [9] We can look at right characteristic sequences and the same will be true for left. So, suppose $\widetilde{u}$ is a right characteristic sequence. Then, as $u_i\psi = u_i\Gamma = u_{i+1}$, $\forall i$, with $u_i \in \widetilde{u}$, we find

$$(u_i\phi)\phi^{-1}\psi\phi = u_i\psi\phi = u_i\Gamma\phi = u_{i+1}\phi,\ \forall i,$$

Hence, $\widetilde{u}\phi$ is a right characteristic sequence. □

We will closely follow the proof of [9, Theorem 9.9] and study the action of $N_G(H)$ on the characteristic sequences.

We will start by proving some lemmas, the first of which gives a finite set of infinite sequences which we will denote by Ω. We then show that we can associate this set Ω to a regular infinite element ψ of H and in fact to H itself. Finally, we let elements of $N_{G_{2,1}}(H)$ act on this finite set of infinite sequences and study the outcome.

Definition 66. *We say that a characteristic sequence $\widetilde{u}$ for H is*

- *fixed set-wise but not pointwise by the element φ if $\widetilde{u}\varphi = \widetilde{u}$ but $u_i\varphi \neq u_i$ for all $u_i \in \widetilde{u}$.*
- *contained in another characteristic sequence $\widetilde{v}$, written $\widetilde{u} \subseteq \widetilde{v}$, if it is contained in $\widetilde{v}$ as a subsequence.*
- *maximal if there exists no element ψ of H with a characteristic sequence $\widetilde{v} \neq \widetilde{u}$ such that $\widetilde{u} \subseteq \widetilde{v}$.*

Lemma 67. *[9] If $\widetilde{u}$ a characteristic sequence of $\psi \in H$ then there is a unique maximal characteristic sequence containing $\widetilde{u}$ as a subsequence.*

For each $\psi \in H$ there is only a finite set of maximal characteristic sequences each fixed set-wise but not point-wise by some power of ψ.

Proof. [9] Suppose without loss of generality that $\widetilde{u}$ is a right characteristic sequence for $\psi \in H$ with $u_i = u\psi^i$ and $u\psi = u\Gamma$ for some non-trivial $\Gamma \in \langle A \rangle$.

Suppose there exists a right characteristic sequence $\widetilde{w}$ for $\varphi \in H$ containing $\widetilde{u}$ as a subsequence *i.e.* $\widetilde{u} \subseteq \widetilde{w}$,

$$u\varphi = u\Delta \text{ and } w_i = u\varphi^i \quad \text{for some non-trivial } \Delta \in \langle A \rangle.$$

As $\widetilde{u} \subseteq \widetilde{w}$ then for some power m of φ we have

$$u\Delta^m = u\varphi^m = u\psi = u\Gamma.$$

Thus, Γ is a proper power of Δ and so $u_{i+1} = u_i\Gamma = u_i\Delta^m = u_i\varphi^m$ for all i. Therefore, there are finitely many right characteristic sequences that contain $\widetilde{u}$ as a subsequence.

Next suppose that there exists $\widetilde{v}$ a characteristic sequence for $\theta \in H$ containing $\widetilde{u}$ as a subsequence with $u\theta = u\Sigma$, $v_i = u\theta^i$ and $\widetilde{v} \neq \widetilde{w}$, $\Sigma \in \langle A \rangle$. If no such $\widetilde{v}$ and $\theta \in H$ exists, then $\widetilde{w}$ is a maximal characteristic sequence. If $\widetilde{v}$ does exist, then given that H is abelian,

$$u\Sigma\Delta = u\theta\varphi = u\varphi\theta = u\Delta\Sigma.$$

Thus, for some word $\Lambda \in \langle A \rangle$ we have $\Delta = \Lambda^a, \Sigma = \Lambda^b$ where a, b are positive integers. Replacing Λ by a power Λ^e if necessary, we may assume that a and b are relatively prime *i.e.* $ac + bd = 1$. Therefore, $u\varphi^c\theta^d = u\Lambda$ and we define a characteristic sequence $\widetilde{W}$ for H by,

$$W_i = u(\varphi^c\theta^d)^i = u\Lambda^i.$$

Hence, $\widetilde{W}$ contains $\widetilde{w}$ and since Γ has only a finite number of initial segments this process terminates in a unique maximal right characteristic sequence for H containing $\widetilde{u}$ as a subsequence.

By Lemma 22 a maximal characteristic sequence contains a semi-infinite orbit and by Lemma 19 there are only finitely many semi-infinite orbits in $V_{2,1}$ under the action of $\varphi^c\theta^d$. Thus, for any element ψ of H there are only a finite number of maximal characteristic sequences each fixed set-wise but not point-wise by some power of ψ. □

Lemma 68. *[9] Let $\widetilde{v}$ be any maximal characteristic sequence for $H = \langle \theta_1, ..., \theta_k \rangle$. Then there are intergers i and m such that θ_i^m fixes the maximal characteristic sequence $\widetilde{v}$ set-wise but not point-wise.*

Proof. [9] If $\widetilde{v}$ is a characteristic sequence for $\psi \in H$ with $v_j\psi = v_j\Gamma = v_{j+1}$ for some non-trivial $\Gamma \in \langle A \rangle$ and if $\varphi \in C_{G_{2,1}}(H)$, in particular if we take $\varphi \in H$ then $\widetilde{v}\varphi$ is one of a finite number of characteristic sequences fixed set-wise but not point-wise by ψ.

Thus, H has a subgroup $\mathcal{B}$ of finite index, m say, fixing $\widetilde{v}$ set-wise. In particular, θ_i^m fixes $\widetilde{v}$ set-wise for each i.

As H is finitely generated by $\theta_1, ..., \theta_k$, since $\psi = \theta_1^{a_1}...\theta_k^{a_k}$ then $\psi^m = \theta_1^{a_1 m}...\theta_k^{a_k m}$. Since, $\psi^m = \theta_1^{a_1 m}...\theta_k^{a_k m}$ does not fix $\widetilde{v}$ point-wise *i.e.* $v_j\psi^m \neq v_j$, then one $\theta_i^{a_i m}$ must not fix $\widetilde{v}$ point-wise. □

We can now see that each element of H has a finite set of maximal characteristic sequences associated to it by Lemma 67. Further, we see that each maximal characteristic sequence associated to an element in H is a characteristic sequence for some power of a generator.

As H is finitely generated, we thus get the number of maximal characteristic sequence that can be associated to H is finite.

Lemma 69. *[9] There are only finitely many maximal characteristic sequences for H.*

Proof. Putting Lemma 67 and Lemma 68 we see that this implies there are in total only finitely many maximal characteristic sequences for H. □

We see that the maximal characteristic sequences are not just associated to the element ψ we have constructed from H, but to H and thus the generators of H.

Definition 70. *We define Ω to be the set of maximal characteristic sequences for H.*

Lemma 71. *[9] Let $\varphi \in N_{G_{2,1}}(H)$. If φ fixes a maximal characteristic sequence $\widetilde{u}$ for H set-wise then for all $u_i \in \widetilde{u}$ we have $u_i\varphi = u_{\eta i+t}$ where $t \in \mathbb{Z}$ and $\eta = \pm 1$.*

Proof. [9] Let $\widetilde{u}$ be a maximal characteristic sequence for $\theta \in H$ and let $\varphi \in N_{G_{2,1}}(H)$. Suppose that $u_0\varphi = u_t$. Then

$$u_i\varphi = u_0\theta^i\varphi = u_0\varphi(\varphi^{-1}\theta\varphi)^i = u_t(\varphi^{-1}\theta\varphi)^i = u_t\psi^i,$$

where $\psi = (\varphi^{-1}\theta\varphi)$. As $\varphi \in N_{G_{2,1}}(H)$, ψ commutes with θ.

If $u_t\psi = u_r$, then for all s

$$u_{r+s} = u_r\theta^s = u_t\psi\theta^s = u_t\theta^s\psi = u_{t+s}\psi.$$

Now, suppose that

$$u_t\psi = u_{t+\eta}.$$

If $u_t\psi^{-1} = u_q$, then

$$u_t = u_t\psi^{-1}\psi = u_q\psi = u_{q+\eta}.$$

So $u_t = u_{q+\eta}$ and thus $q = -\eta + t$ and $u_t\psi^{-1} = u_{-\eta+t}$. Inductively, we have for all integers i, $u_i\varphi = u_t\psi^i = u_{\eta i+t}$.

If $|\eta| \neq 1$ then φ maps $\widetilde{u}$ into a proper subset so $\widetilde{u} \supset \widetilde{u}\varphi$. However, $\widetilde{u}$ is a maximal characteristic sequence and so by applying φ^{-1} to both sides, we get $\widetilde{u}\varphi^{-1} \supset \widetilde{u}$, which is a characteristic sequence for $\varphi\theta\varphi^{-1} \in H$. This contradicts the maximal nature of $\widetilde{u}$. Thus $|\eta| = 1$. □

The action of $N_{G_{2,1}}(H)$ on the maximal characteristic sequences for H allows us to consider a new set of *oriented* maximal characteristic sequences Ω'.

That is, there are elements of $N_{G_{2,1}}(H)$ that act by conjugation which preserve the orientation of the maximal characteristic sequences by mapping $\widetilde{v} \mapsto \widetilde{u}$ for $\widetilde{u}, \widetilde{v} \in \Omega$ and maps that reverse the orientation of a maximal characteristic sequence $\widetilde{v} \mapsto \widetilde{u}^R$ for $\widetilde{u}, \widetilde{v} \in \Omega$, that is $\widetilde{u}$ but with the direction reversed.

Definition 72. *We define Ω' to be the set of oriented maximal characteristic sequences for H, that is*

$$\Omega' = \{\widetilde{u}, \widetilde{u}^R | \widetilde{u} \in \Omega\}.$$

Lemma 73. *[9] Let H be a finitely generated free-abelian subgroup of $G_{2,1}$, then there exists subgroups M and L of $N_{G_{2,1}}(H)$ such that,*

(1) M is a normal subgroup of finite index in $N_{G_{2,1}}(H)$;
(2) and L is a normal subgroup of M where $L \cap H = \{Id_{G_{2,1}}\}$ and M/L is a free abelian group of finite rank.

Proof. [9] *(1)* Firstly, we define the set Ω' of all maximal characteristic sequences and their reverses *i.e.* all oriented maximal characteristic sequences.

For $\varphi \in N_{G_{2,1}}(H)$ and a maximal characteristic sequence $\widetilde{u} = (u_i)$ we have

$$\varphi : (u_i) \to (u_i\varphi).$$

Let M be the kernel of the action of $N_{G_{2,1}}(H)$ on the finite set Ω'. Thus, M is a normal subgroup of finite index in $N_{G_{2,1}}(H)$.

(2) Secondly, an element ϕ in M acts on the finite set of oriented maximal characteristic sequences $\Omega^{'}$ as

$$v_j^{(i)} \mapsto v_{j+t_i}^{(i)}.$$

If we let the oriented maximal characteristic sequences be $\widetilde{v}^{(1)}, ..., \widetilde{v}^{(m)}$ and the t_i's corresponding to an element ϕ of M be $t_1, ..., t_m$, then the map

$$\Psi : M \to \mathbb{Z}^m,$$

where $\phi \mapsto (t_1, ..., t_m)$ is a surjective homomorphism of M into the free abelian group of finite rank. The kernel of Ψ, L, is the set of elements of M that fix each oriented maximal characteristic sequence point-wise. L is a normal subgroup of M.

Let ϑ be a non-trivial element of H in semi-normal form with respect to X. Since ϑ has infinite order, by Theorem 24 some proper power of ϑ has a proper characteristic multiplier Γ say. Let $v_0 \in X$ be a characteristic element for ϑ^n, so

$$v_0\vartheta^n = v_0\Gamma = v_1 \text{ and } v_0\vartheta^{ni} = v_i.$$

Then

$$v_i\vartheta^n = v_0\vartheta^{ni}\vartheta^n = v_0\vartheta^n\vartheta^{ni} = v_0\Gamma\vartheta^{ni} = v_0\vartheta^{ni}\Gamma = v_i\Gamma \neq v_i.$$

Therefore $\vartheta^n \notin L$ and thus $\vartheta \notin L$, $H \cap L = \{Id_{G_{2,1}}\}$.

We thus have the (sub-)normal series,

$$N \triangleright M \triangleright L \triangleright 1,$$

with N/M finite, M/L free abelian of finite rank and $L \cap H = 1$. □

From Lemma 73 we now have a description of the elements of $N_{G_{2,1}}(H)$ in terms of the subgroups M and L.

In fact we have two sequences, given by the proof of Lemma 73. The first sequence

$$1 \to M \to N_{G_{2,1}}(H) \to Sym(\Omega') \to 1,$$

given by the action of elements of $N_{G_{2,1}}(H)$ on the orientated maximal characteristic sequences for H, where $Sym(\Omega')$ is a permutation group of the orientated maximal characteristic sequences for H. We can define this by the map $\gamma : N_{G_{2,1}}(H) \to Sym(\Omega')$, where $\gamma(\varphi) = \{\widetilde{u}^{(1)}, ..., \widetilde{u}^{(m)}\}\varphi = \{\widetilde{u}^{(1)}\varphi, ..., \widetilde{u}^{(m)}\varphi\}$ for $\varphi \in N_{G_{2,1}}(H)$ and $\{\widetilde{u}^{(1)}, ..., \widetilde{u}^{(m)}\} = \Omega'$.

Secondly,

$$1 \to L \to M \to \mathbb{Z}^m \to 1,$$

given by the map Ψ in Lemma 73, where L is a normal subgroup of M. We are now ready to prove the Theorem 62.

Proof of Theorem 62:[9] By Lemma 73, since $H \cap L = 1$, then using the third isomorphism theorem,

$$(H \cap M)L/L \cong (H \cap M)/(H \cap M \cap L) \cong H \cap M.$$

As

$$(H \cap M)L/L \subseteq M/L,$$

is free abelian, then $H \cap M$ is free abelian. Since

$$H/H \cap M \cong HM/M \subseteq N_{G_{2,1}}(H)/M,$$

is finite, then $H \cap M$ is of finite index in H. There is an n such that $H \cap M \geq H^n = \{h^n | h \in H\}$ and H^n is torsion-free, as it is finitely generated and free-abelian, so we must have $H \cong H^n$. We have that $H^n \leq H \cap M \leq M$. Now, since H is abelian and contains the commutator $[H, M]$ (as $M \subseteq N_{G_{2,1}}(H)$), we can show that $[H, M]^n = [H^n, M]$. In fact we can show that $\forall h \in H$ and $m \in M$

$$[h, m]^n = [h^n, m].$$

Let $h, h' \in H$ and $m \in M$ then

$$[hh', m] = [h, m]^{h'}[h', m],$$

by a standard commutator identity, see [14]. Now, as

$$[h, m]^{h'}[h', m] = [h, m][h', m].$$

Since

$$[h^k, m] = [h^{k-1}h, m] = [h^{k-1}, m][h, m],$$

we get for $k = n$,

$$[h^n, m] = [h, m]^n.$$

We can now us this to show that actually $\forall m \in M$, m centralizes H.

$$[H,M]^n = [H^n,M] \leq [H \cap M, M] \leq [H,M] \cap [M,M]$$
$$\leq [N_{G_{2,1}}(H), H] \cap [M,M] \leq H \cap L = \{Id_{G_{2,1}}\},$$

and since $[H,M]$ is torsion-free, $[H,M] = 1$. Thus $M \leq C_{G_{2,1}}(H)$, whence $C_{G_{2,1}}(H)$ has finite index in $N_{G_{2,1}}$. □

6.2. *Centralizing regular infinite elements*

We will, in general, refer to the regular infinite part of an element of $G_{2,1}$ as a regular infinite element. This is an abuse of Definition 29. However, as the regular infinite part of an element ψ acts on the subalgebra $V_{RI,\psi}$ as an automorphism of the subalgebra we are not being too precise. This inevitably saves cumbersome notation and language in the proceeding subsections.

We will now introduce a partition of the subalgebra $Y\langle A\rangle$ induced by a regular infinite element ψ. This in turn gives way to a decomposition of ψ in which the disjoint semi-infinite orbits of ψ have elements of the orbit in disjoint sets of the subalgebra $Y\langle A\rangle$.

Definition 74. *Let ψ be a regular infinite element and in quasi-normal form with respect to the basis Y. We can decompose Y (and thus $Y\langle A\rangle$) as,*

$$Y = \coprod_{i=1}^{k} Y_i.$$

where Y_i is defined by the intersection of Y with one of the equivalence classes defined in Definition 36. Then $\psi = \psi_1 ... \psi_k$ where ψ_i is defined on Y by

$$y\psi_i = \begin{cases} y\psi & \textit{if } y \in Y_i, \\ y & \textit{if } y \in Y_j \textit{ for } i \neq j, \end{cases}$$

and we call Y_i the support of ψ_i. Then $\psi_i \psi_j = \psi_j \psi_i$.

Remark 75. This is similar to the decomposition in [3] given by the disjoint flow graph components of a given infinite order element of $G_{2,1}$.

Remark 76. Let ψ, φ be regular infinite elements of $G_{2,1}$ with respect to the bases X, Y. By Proposition 43, we can construct a finite set of maps $\mathcal{R}(\psi;\varphi)$ of X into $Y\langle A\rangle\langle\lambda\rangle$ such that if ψ is conjugate to φ, then a map $\rho_0 \in \mathcal{R}(\psi;\varphi)$ extend to an isomorphism

$$\rho : X\langle A\rangle\langle\lambda\rangle \to Y\langle A\rangle\langle\lambda\rangle.$$

We constructed such maps by looking at a least equivalence relation on the basis X, see Definition 36. The equivalence classes constructed are preserved by the decomposition of the basis as defined in Definition 74. The maps from $\mathcal{R}(\psi;\varphi)$ map one equivalence class of $X\langle A\rangle$ to one of $Y\langle A\rangle$ with the same characteristic multipliers (see Proposition 37). If we are looking at centralizing elements g (*i.e.* $\psi = \varphi$ and $X = Y$) then

$$g^{-1}\psi g = g^{-1}\psi_1 g....g^{-1}\psi_k g,$$

where each $g^{-1}\psi_i g$ is centralized by g or conjugated to a ψ_j (*i.e.* g permutes the decomposition) by the above argument.

Lemma 77. *Let ψ be a regular infinite element of $G_{2,1}$ then $\psi = \psi_1...\psi_k$ where the ψ_i's are regular infinite elements and $H = \langle \psi_1, ..., \psi_k \rangle$ is a finitely generated free abelian subgroup of $G_{2,1}$.*

Proof. By Definition 74 ψ can be decomposed as $\psi_1...\psi_k$, where $\psi_i\psi_j = \psi_j\psi_i$ and ψ_i are regular infinite for all i, j. Therefore, H is a finitely generated free abelian subgroup of $G_{2,1}$. □

Theorem 78. *Let ψ be a regular infinite element of $G_{2,1}$ then there exists $H = \langle \psi_1, ..., \psi_k \rangle$ a finitely generated torsion-free abelian subgroup of $G_{2,1}$ such that*

$$C_{G_{2,1}}(H) \subseteq C_{G_{2,1}}(\psi) \subseteq N_{G_{2,1}}(H).$$

Therefore, $C_{G_{2,1}}(\psi)$ has finite index in $N_{G_{2,1}}(H)$.

Proof. By Lemma 77 for any regular infinite element ψ we can construct a finitely generated torsion-free abelian subgroup of $G_{2,1}$, H and from Theorem 62 we know that $C_{G_{2,1}}(H)$ has finite index in $N_{G_{2,1}}(H)$.

Obviously, $C_{G_{2,1}}(H) \subseteq C_{G_{2,1}}(\psi)$. Given any $g \in C_{G_{2,1}}(\psi)$ by Remark 76 we see that

$$g^{-1}\psi g = g^{-1}\psi_1 g....g^{-1}\psi_k g,$$

where each $g^{-1}\psi_i g$ is centralized by g or conjugated by g to a ψ_j. Thus, $g \in N_{G_{2,1}}(H)$ and hence $C_{G_{2,1}}(H) \subseteq C_{G_{2,1}}(\psi) \subseteq N_{G_{2,1}}(H)$.

Therefore, $C_{G_{2,1}}(\psi)$ has finite index in $N_{G_{2,1}}(H)$. □

Remark 79. As we have chosen a very special H, we can say that L will just consist of the identity element in $G_{2,1}$. As the only element that fixes characteristic sequences for regular infinite elements in this case is the identity.

Hence, as the proof of Theorem 62 describes elements in $C_{G_{2,1}}(H)$ and $N_{G_{2,1}}(H)$ then we understand the elements in $C_{G_{2,1}}(\psi)$ for ψ a regular infinite element. We can therefore form the following corollary.

Corollary 80. *Let ψ be a regular infinite element of $G_{2,1}$ in quasi-normal form with respect to Y. If ψ has Ω' as its set of oriented maximal characteristic sequences, then*

$$1 \to \mathbb{Z}^k \to C_{G_{2,1}}(\psi) \to S \to 1,$$

where $S \subseteq Sym(\Omega')$ and k is given by Definition 36.

Proof. Let $\psi = \psi_1 ... \psi_k$ be in quasi-normal form with respect to Y (partitioned according to Definition 36) and $\Omega^{'}$ the set of oriented maximal characteristic sequences for $H = \langle \psi_1, ..., \psi_k \rangle$. If $|\Omega^{'}| = m$, we can replace $\mathbb{Z}^m$ by $\mathbb{Z}^k$ because the oriented characteristic sequences will intersect the k equivalence classes, by Definition 36.

As any element that centralizes ψ will permute the decomposition $\psi_1 ... \psi_k$ (see Remark 76), we thus see that the map $\Psi : M \to \mathbb{Z}^m$ (Lemma 73) is actually a map to $\mathbb{Z}^k$.

More specifically, the partition of Y induced by the equivalence relation from Definition 36 also partitions up the set of oriented maximal characteristic sequences for H. That is, we can group the $t_i^{\prime}s$ corresponding to which oriented maximal characteristic sequences for H intersect each of the equivalence classes of elements of Y. □

6.3. *Centralizing periodic elements*

We will, in general, refer to the periodic part of an element of $G_{2,1}$ as a periodic element. This is an abuse of the Definition 30. However, as the periodic part of an element ψ acts on the subalgebra $V_{P,\psi}$ as an automorphism of the subalgebra we are not being too precise. This inevitably saves cumbersome notation and language in the proceeding subsections.

We can consider each periodic element ψ as being defined by a basis Y (for which ψ is in quasi-normal form) and a permutation σ of that basis, that is an element in $Sym(Y)$ with $|Y| = m$. All orbits of the subalgebra $Y\langle A \rangle$ will be an orbit of the same cycle type as that in σ.

This can be seen in Graham Higman's work. Looking at Lemma 22 the basis giving a finite order element of $G_{2,1}$ in quasi-normal form completely determines the finite orbits. Therefore, for each periodic element ψ of $G_{2,1}$ it is sufficient to look at a basis Y for which ψ is in quasi-normal form and a permutation of that basis corresponding to the orbit structure of ψ for elements of Y.

We note that if Y is a basis giving a periodic element ψ in quasi-normal form, then any uniform m-fold expansion on Y ($|Y| = m$) gives ψ in semi-normal form with respect to the new basis.

Lemma 81. *Let ψ be a torsion element of $G_{2,1}$ of order n in quasi-normal form with respect to the basis $X = \{x_0, ..., x_{n-1}\}$ such that $x_i\psi = x_{i+1}$ taking the indices modulo n. Let ρ be an element of $G_{2,1}$.*

Then, ρ centralizes ψ if and only for some $m \in \mathbb{N}$ we have bases of the form

$$Y = \{x_0\Gamma_1, ..., x_0\Gamma_m, ..., x_{n-1}\Gamma_1, ..., x_{n-1}\Gamma_m\}$$

and

$$Z = \{x_0\Delta_1, ..., x_0\Delta_m, ..., x_{n-1}\Delta_1, .., x_{n-1}\Delta_m\}$$

such that for a chosen set $k_j \in \{0, ..., n-1\}$ *for* $j = 1, ..., m$ *and* $\tau \in Sym(m)$*,* ρ *is defined by the equation,*

$$x_i \Gamma_j \rho = x_{i+k_j} \Delta_{\tau(j)},$$

where $\Gamma_j, \Delta_j \in \langle A \rangle$ *and* $i + k_j$ *is taken modulo* n*.*

Proof. Firstly, let

$$Y = \{x_0\Gamma_1, ..., x_0\Gamma_m, ..., x_{n-1}\Gamma_1, ..., x_{n-1}\Gamma_m\}$$

and

$$Z = \{x_0\Delta_1, ..., x_0\Delta_m, ..., x_{n-1}\Delta_1, .., x_{n-1}\Delta_m\}$$

with ρ defined by the equation

$$x_i \Gamma_j \rho = x_{i+k_j} \Delta_{\tau(j)},$$

for chosen $k_1, ..., k_m$ and $\tau \in Sym(m)$.

Thus,

$$x_i \Gamma_j \rho \psi = x_{i+k_j} \Delta_{\tau(j)} \psi = x_{i+k_j+1} \Delta_{\tau(j)},$$

and

$$x_i \Gamma_j \psi \rho = x_{i+1} \Gamma_j \rho = x_{i+k_j+1} \Delta_{\tau(j)},$$

where $i + k_j + 1$ is taken modulo n. Thus $\psi\rho = \rho\psi$.

Secondly, assume ρ centralizers ψ. Let $\rho : Y \to Z$ where $Y, Z \subseteq X\langle A \rangle$. We will show that we can choose Y, Z to be of the form given in this lemma.

Let ρ be in quasi-normal form with respect to Y. By Lemma 27 this basis is unique. As ρ centralizes ψ, $\rho : Y\psi \to Z\psi$. Since $x_i \psi^a = x_{i+a}$ (indices taken modulo n), ψ fixes Y setwise and thus Y has the form $\{x_i \Gamma_j\}$, for $x_i \in X$ and $\Gamma_j \in \langle A \rangle$. Since

$$Z\psi = Y\rho\psi = Y\psi\rho = Y\rho = Z,$$

we see that Z is fixed setwise by ψ and thus has the form $\{x_i \Delta_j\}$, for $x_i \in X$ and $\Delta_j \in \langle A \rangle$. We can therefore define ρ as the map,

$$x_i \Gamma_j \rho = x_{F(i,j)} \Delta_{G(i,j)}$$

for $x_i \Gamma_j \in Y$, $x_{F(i,j)} \Delta_{G_{(i,j)}} \in Z$, where F, G are functions that might depend on i and j.

Now, as ρ conjugates ψ to φ,

$$x_i \Gamma_j \rho \psi = x_{F(i,j)} \Delta_{G(i,j)} \psi = x_{F(i,j)+1} \Delta_{G(i,j)},$$

and

$$x_i \Gamma_j \psi \rho = x_{i+1} \Gamma_j \rho = x_{F(i+1,j)} \Delta_{G(i+1,j)}.$$

As $\psi\rho = \rho\psi$ we have $F(i,j) + 1 = F(i+1,j)$ and $G(i+1,j) = G(i,j)$. Therefore, inductively on i we have $F(i,j) = i + k_j$ where k_j is some function that depends on j and can take the values in the set $\{0, ..., n-1\}$. As $G(i,j) = G(i+1,j)$ the function G is independent of i, *i.e.* there exists $\tau \in Sym(m)$ such that $G(i,j) = \tau(j)$. □

We next show that we can conjugate our torsion element ψ given in quasi-normal form with respect to X with r cycles of length s to an element ψ^g with one single cycle of length s on a new basis X' and thus we need only to consider different sized cycles in a permutation of a quasi-normal form basis for a torsion element of $G_{2,1}$.

The following lemma follows from the work of [9, Section 6].

Lemma 82. *Let ψ, φ be torsion elements of $G_{2,1}$. Suppose that ψ is in quasi-normal form with respect to X of size rn, φ is in quasi-normal form with respect to Y of size n and that each element has cycles of length n with respect to those bases. Then there exists a map ρ from X to a proper expansion Y' of Y that is an element of $G_{2,1}$ such that $\rho^{-1}\psi\rho = \varphi$.*

Proof. Without loss of generality, let ψ be in quasi-normal form with respect to the basis

$$X = \{x_0, ..., x_{n-1}, ..., x_{rn-n}, ..., x_{nr-1}\},$$

and defined by $x_{in+j}\psi = x_{in+(j+1)_{\text{mod } n}}$ for $i = 0, ..., r-1$ and $j = 0, ..., n-1$ where $(j+1)_{\text{mod } n}$ means take $j+1$ mod n.

Let φ be in quasi-normal form with respect to the basis $Y = \{y_0, ..., y_{n-1}\}$ and defined by $y_k\varphi = y_{(k+1)_{\text{mod } n}}$ for $k = 0, ..., n-1$

Let Y' be a basis of $V_{2,1}$ which is a proper expansion of Y of the form $\{y_j\Gamma_i\}$ for $i = 0, ..., n-1$, $j = 0, ..., n-1$ and $\Gamma_i \in \langle A \rangle$. Define a map ρ by:

$$x_{in+j} \mapsto y_j\Gamma_i,$$

for $j = 0, ..., n-1$, $i = 0, ..., r-1$. Thus, ρ is a bijective map

$$\rho : X \to Y',$$

of bases, where $|X| = |Y'| = rn$ and so $\rho \in G_{2,1}$.

We now check whether the equation $\psi\rho = \rho\varphi$ holds.

(1) We take the basis X and apply ψ then ρ. Thus, we have

$$x_{in+j}\psi\rho = x_{in+(j+1)_{\text{mod } n}}\rho = y_{(j+1)_{\text{mod } n}}\Gamma_i,$$

for $j = 0, ..., n-1$ and $i = 0, ..., r-1$.

(2) We take the basis X and apply ρ then φ. Thus, we have

$$x_{in+j}\rho\varphi = y_j\Gamma_i\varphi = y_j\varphi\Gamma_i = y_{(j+1)_{\text{mod } n}}\Gamma_i,$$

for $j = 0, ..., n-1$ and $i = 0, ..., r-1$.

Hence, ρ conjugates ψ to φ. □

We therefore get the following consequence of Lemma 82.

Corollary 83. *Let ψ, φ be torsion elements of $G_{2,1}$ as described in Lemma 82. Then $C_{G_{2,1}}(\psi) \cong C_{G_{2,1}}(\varphi)$.*

Proof. From the proof of Lemma 82 ψ and φ are conjugate by an element of $G_{2,1}$, ρ. Therefore, there exists a map $f : C_{G_{2,1}}(\psi) \to C_{G_{2,1}}(\varphi)$ defined by $f(g) = g^{\rho}$ for all $g \in C_{G_{2,1}}(\psi)$. This is an isomorphism and so $C_{G_{2,1}}(\psi) \cong C_{G_{2,1}}(\varphi)$. □

From the solution to the conjugacy problem in $G_{n,r}$ for $n \geq 2$ and $r \in \mathbb{N}$ we can not mix cycles of different lengths (see Proposition 34). We therefore get the following lemma.

Lemma 84. *Suppose ψ is a torsion element of $G_{2,1}$ in quasi-normal form with respect to the basis X with r_i cycles of length s_i for $i = 1, ..., k$ on the basis X and φ a torsion element of $G_{2,1}$ in quasi-normal form with respect to Y with one cycle of length s_i for $i = 1, ..., k$ on the basis Y. Then, there exists an element ρ of $G_{2,1}$ such that $\rho^{-1}\psi\rho = \varphi$. Furthermore, $C_{G_{2,1}}(\psi) \cong C_{G_{2,1}}(\varphi)$.*

Proof. Firstly, by the solution to the conjugacy problem we can not mix different cycles of different lengths. Therefore, we can apply a variation of Lemma 82 to each portion of the basis X where r_i-cycles of length s_i are defined for each i. Hence we define a map ρ between the basis X and a proper expansion Y' of the basis Y such that $\rho^{-1}\psi\rho = \varphi$. Consequently, by a variation of Corollary 83, $C_{G_{2,1}}(\psi) \cong C_{G_{2,1}}(\varphi)$. □

We thus only need to consider torsion elements of $G_{2,1}$ with one cycle of each cycle type present in a basis for that given element. Therefore, a generalization of Lemma 81 follows from the solution to the conjugacy problem, *i.e.* n-cycles and m-cycles are not conjugate for $n \neq m$. Hence, centralizing elements centralize each n-cycle and m-cycle separately.

Theorem 85. *Let ψ be a torsion element of $G_{2,1}$ in quasi-normal form with respect to the basis X with ψ having one cycle of length s_i on the basis X for $i = 1, ..., n$.*

Then, ρ centralizes ψ if and only if for some $m_{s_i} \in \mathbb{N}$ we have bases of the form

$$\begin{aligned} Y = \{ & x_0^{(1)}\Gamma_{1_{s_1}}, ..., x_0^{(1)}\Gamma_{m_{s_1}}, ..., x_{s_1-1}^{(1)}\Gamma_{1_{s_1}}, ..., x_{s_1-1}^{(1)}\Gamma_{m_{s_1}}, \\ & x_0^{(2)}\Gamma_{1_{s_2}}, ..., x_0^{(2)}\Gamma_{m_{s_2}}, ..., x_{s_2-1}^{(2)}\Gamma_{1_{s_2}}, ..., x_{s_2-1}^{(2)}\Gamma_{m_{s_2}}, ..., \\ & x_0^{(n)}\Gamma_{1_{s_n}}, ..., x_0^{(n)}\Gamma_{m_{s_n}}, ..., x_{s_n-1}^{(n)}\Gamma_{1_{s_n}}, ..., x_{s_n-1}^{(n)}\Gamma_{m_{s_n}} \} \end{aligned}$$

and

$$Z = \{ x_0^{(1)}\Delta_{1_{s_1}}, ..., x_0^{(1)}\Delta_{m_{s_1}}, ..., x_{s_1-1}^{(1)}\Delta_{1_{s_1}}, ..., x_{s_1-1}^{(1)}\Delta_{m_{s_1}}, ...,$$

$$x_0^{(2)}\Delta_{1_{s_2}},...,x_0^{(2)}\Delta_{m_{s_2}},...,x_{s_2-1}^{(2)}\Delta_{1_{s_2}},...,x_{s_2-1}^{(2)}\Delta_{m_{s_2}},...,$$
$$x_0^{(n)}\Delta_{1_{s_n}},...,x_0^{(n)}\Delta_{m_{s_n}},...,x_{s_n-1}^{(n)}\Delta_{1_{s_n}},...,x_{s_n-1}^{(n)}\Delta_{m_{s_n}}\}$$

such that for a chosen set $k_l^{(i)} \in \{0,...,s_i-1\}$ for $l = 1_{s_i},...,m_{s_i}$ and $\tau^{(i)} \in Sym(m_{s_i})$, ρ is defined by the equation

$$x_j^{(i)}\Gamma_l^{(i)}\rho = x_{j+k_l^{(i)}}^{(i)}\Delta_{\tau^{(i)}(l)}^{(i)},$$

where $\Gamma_l^{(i)}, \Delta_l^{(i)} \in \langle A\rangle$ and $j+k_l^{(i)}$ is taken modulo s_i for $i=1,...,n$ and $j=0,...,s_i-1$.

Proof. From the solution to the conjugacy problem for $G_{2,1}$ we see that centralizing each s_i-cycle is disjoint form centralizing each s_j-cycle for $i \neq j$. We can therefore apply Lemma 81 to each s_i-cycle for $i=1,...,n$. The result then follows. □

This completely describes centralizing elements for torsion elements of $G_{2,1}$. We can now go further if we consider a splitting of a periodic element ψ of $G_{2,1}$.

Definition 86. *Let ψ be a torsion element of $G_{2,1}$ in quasi-normal form with respect to the basis Y with ψ having one cycle of length s_i on the basis Y for $i=1,...,n$. We can decompose Y (and thus $Y\langle A\rangle$) as,*

$$Y = \coprod_{i=1}^{n} Y_i.$$

where Y_i is defined by the cycle of length s_i. Then $\psi = \psi_1...\psi_n$ where ψ_i is defined on Y by

$$y\psi_i = \begin{cases} y\psi & \text{if } y \in Y_i, \\ y & \text{if } y \in Y_j \text{ for } i \neq j, \end{cases}$$

and we call Y_i the support of ψ_i. Then $\psi_i\psi_j = \psi_j\psi_i$.

We can thus form the subgroup for ψ with respect to the decomposition given in Definition 86, $H = \langle\psi_1,...,\psi_n\rangle$. Obviously, $H \cong C_{s_1} \times .... \times C_{s_n}$, where C_{s_i} is the cyclic group of order s_i and $H \subseteq C_{G_{2,1}}(\psi)$.

Firstly, we define a map $\Psi_i : C_{G_{2,1}}(\psi_i)|_{Y_i} \to G_{2,1}$ in the following way. For $\varphi_i \in C_{G_{2,1}}(\psi_i)|_{Y_i}$

$$\varphi_i \mapsto \Psi_i(\varphi_i),$$

where $\Psi_i(\varphi_i)$ is defined by the bases $Y = \{x\Gamma_{1_{s_i}},...,x\Gamma_{m_{s_i}}\}$, $Z = \{x\Delta_{1_{s_i}},...,x\Delta_{m_{s_i}}\}$ and the map

$$x\Gamma_{j_{s_i}}\Psi_i(\varphi_i) = x\Delta_{\tau(j_{s_i})},$$

where $\tau \in Sym(m_{s_i})$. This is obviously an element of $G_{2,1}$ as we have formed a bijective map between two bases. Secondly, the map is a surjective homomorphism of $C_{G_{2,1}}(\psi_i)|_{Y_i} \to G_{2,1}$.

We can see that Ψ_i is a homomorphism by looking at a torsion element with a single cycle of length n on some basis.

Lemma 87. *The map Ψ_i is a homomorphism.*

Proof. Let ψ be a periodic element of $G_{2,1}$ with a single cycle of length n on the basis $X = \{x_0, ..., x_{n-1}\}$. Let $\varphi_1, \varphi_2 \in C_{G_{2,1}}(\psi)$.

Let φ_1 be defined by $m_1 \in \mathbb{N}$ with bases of the form

$$Y_1 = \{x_0\Gamma_1, ..., x_0\Gamma_m, ..., x_{n-1}\Gamma_1, ..., x_{n-1}\Gamma_m\}$$

and

$$Z_1 = \{x_0\Delta_1, ..., x_0\Delta_m, ..., x_{n-1}\Delta_1, .., x_{n-1}\Delta_m\}$$

for a chosen set $k_j^{(1)} \in \{0, ..., n-1\}$ for $j = 1, ..., m_1$ and $\tau_1 \in Sym(m_1)$ with equation,

$$x_i\Gamma_j\varphi_1 = x_{i+k_j^{(1)}}\Delta_{\tau(j)},$$

where $\Gamma_j, \Delta_j \in \langle A \rangle$ and $i + k_j^{(1)}$ is taken modulo n.

Let φ_2 be defined by $m_2 \in \mathbb{N}$ with bases of the form

$$Y_2 = \{x_0\Sigma_1, ..., x_0\Sigma_{m_2}, ..., x_{n-1}\Sigma_1, ..., x_{n-1}\Sigma_{m_2}\}$$

and

$$Z_2 = \{x_0\Xi_1, ..., x_0\Xi_{m_2}, ..., x_{n-1}\Xi_1, .., x_{n-1}\Xi_{m_2}\}$$

for a chosen set $k_l^{(2)} \in \{0, ..., n-1\}$ for $l = 1, ..., m_2$ and $\tau_2 \in Sym(m_2)$ with equation,

$$x_i\Gamma_l\varphi_2 = x_{i+k_l^{(2)}}\Delta_{\tau(l)},$$

where $\Sigma_l, \Xi_l \in \langle A \rangle$ and $i + k_l^{(2)}$ is taken modulo n.

We look at the maps $x_i\Gamma_j\varphi_1 = x_{i+k_j^{(1)}}\Delta_{\tau(j)}$ for $j = 1, .., m_1$ and $x_i\Gamma_l\varphi_2 = x_{i+k_j^{(2)}}\Delta_{\tau(l)}$ for $l = 1, .., m_2$. Under Ψ we get maps $x\Gamma_j\Psi(\varphi_1) = x\Delta_{\tau_1(j)}$ and $x\Sigma_l\Psi(\varphi_2) = x\Delta_{\tau_2(l)}$ (where x is the element in the set $\mathbf{x}$). Thus,

$$x_i\Gamma_j\varphi_1\varphi_2 = x_{i+k_j^{(1)}}\Delta_{\tau(j)}\varphi_2.$$

We can then pass under the map Ψ to $G_{2,1}$ by simply forgetting about the $x_{i+k_j^{(1)}}$ so that we send $x_{i+k_j^{(1)}}\Delta_{\tau(j)}\varphi_2 \mapsto x\Delta_{\tau_1(j)}\Psi(\varphi_2)$. We also have,

$$x\Gamma_j\Psi(\varphi_1)\Psi(\varphi_2) = x\Delta_{\tau_1(j)}\Psi(\varphi_2).$$

Hence, Ψ is a homomorphism. □

We can therefore form the map

$$\Psi : C_{G_{2,1}}(\psi) \to G_{2,1} \times ... \times G_{2,1},$$

where $\Psi = \Psi_1 \circ ... \circ \Psi_n$ is a composition of the maps defined for each i. The kernel of this map is the subgroup G (which contains the subgroup H defined above) consisting of all elements of $\rho \in C_{G_{2,1}}(\psi)$ with $Y\rho = Y$. Where Y is given by,

$$Y = \{x_0^{(1)}\Gamma_{1_{s_1}}, ..., x_0^{(1)}\Gamma_{m_{s_1}}, ..., x_{s_1-1}^{(1)}\Gamma_{1_{s_1}}, ..., x_{s_1-1}^{(1)}\Gamma_{m_{s_1}},$$
$$x_0^{(2)}\Gamma_{1_{s_2}}, ..., x_0^{(2)}\Gamma_{m_{s_2}}, ..., x_{s_2-1}^{(2)}\Gamma_{1_{s_2}}, ..., x_{s_2-1}^{(2)}\Gamma_{m_{s_2}}, ...,$$
$$x_0^{(n)}\Gamma_{1_{s_n}}, ..., x_0^{(n)}\Gamma_{m_{s_n}}, ..., x_{s_n-1}^{(n)}\Gamma_{1_{s_n}}, ..., x_{s_n-1}^{(n)}\Gamma_{m_{s_n}}\}$$

for a chosen set $k_l^{(i)} \in \{0, ..., s_i - 1\}$ for $l = 1_{s_i}, ..., m_{s_i}$, $j = 0, ..., s_i - 1$. Thus, ρ is defined by the equation

$$x_j\Gamma_l\rho = x_{j+k_l^{(i)}}\Gamma_l,$$

where $\Gamma_l \in \langle A \rangle$ and $j + k_l^{(i)}$ is taken modulo s_i for $i = 1, ..., n$.

We therefore form a short exact sequence,

$$1 \to G \to C_{G_{2,1}}(\psi) \to (G_{2,1})^n \to 1.$$

If we form the map $\Phi_i : G_{2,1} \to C_{G_{2,1}}(\psi_i)|_{Y_i}$ for $\varphi_i \in G_{2,1}$,

$$\varphi_i \mapsto \Phi_i(\varphi_i),$$

where $\Phi_i(\varphi_i)$ is defined by the basis

$$Y = \{x_0^{(i)}\Gamma_{1_{s_i}}, ..., x_0^{(i)}\Gamma_{m_{s_i}}, ..., x_{s_i-1}^{(i)}\Gamma_{1_{s_i}}, ..., x_{s_i-1}^{(i)}\Gamma_{m_{s_i}}\}$$

and the basis

$$Z = \{x_0^{(i)}\Delta_{1_{s_i}}, ..., x_0^{(i)}\Delta_{m_{s_i}}, ..., x_{s_i-1}^{(i)}\Delta_{1_{s_i}}, .., x_{s_i-1}^{(i)}\Delta_{m_{s_i}}\}$$

with map

$$x_j^{(i)}\Gamma_k\Phi_i(\varphi_i) = x_j^{(i)}\Delta_{\tau(k)},$$

for each $i = 1, ..., n$ with $\tau \in Sym(m_{s_i})$, $j = 0, ..., s_i - 1$ and $k = 1_{s_i}, ..., m_{s_i}$. We can compose the maps $\Phi_1 \circ ... \circ \Phi_n = \Phi$. The map $\Psi(\Phi((\psi_1, ..., \psi_n))) = (\psi_1, ..., \psi_n)$, thus $\Psi\Phi$ acts as the identity on $(G_{2,1})^n$. Hence, we get a splitting of the short exact sequence so that $C_{G_{2,1}}(\psi) = G \rtimes (G_{2,1})^n$.

As each centralizing element is build up out of the centralizing maps for the s_i cycles on the basis Y, we can describe the subgroup G (which contains $H \cong C_{s_1} \times ... \times C_{s_n}$) as $G = G_{s_1} \times ... \times G_{s_n}$.

Thus we have $C_{G_{2,1}}(\psi) \cong \prod_{i=1}^{n}(G_{s_i} \rtimes G_{2,1})$.

Theorem 88. *Let ψ be a periodic element of $G_{2,1}$. If ψ is in quasi-normal form with respect to the basis Y with ψ having one cycle of length s_i on the basis Y for $i = 1, ..., n$, then*

$$C_{G_{2,1}}(\psi) \cong \prod_{i=1}^{n} (G_{s_i} \rtimes G_{2,1})$$

where the G_{s_i}'s are defined above.

Proof. Follows from the previous work in this section. □

By definition of the kernel of the map Ψ, we can see that the G_{s_i}'s are infinite groups of s_i periodic elements. We would like to know if they are finitely generated.

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References

[1] Anshel, M.; Stebe, P., (1974) The solvability of the conjugacy problem for certain HNN groups, *Bull. Amer. Math. Soc.*, **80** (2) 266–270.
[2] J. Belk and F. Matucci, Conjugacy in Thompson's Groups, http://arxiv.org/abs/0708.4250 (2007).
[3] C. Bleak, A. Gordon, G. Garrett, H. Newfield-Plunkett-Bowman, E. Sapir, F. Matucci and J. Hughes, Centralizers in R.Thompson's group V_n. In preparation (2009).
[4] C. Bleak, M. Kassabov and F. Matucci, Roots, Centralizers and the Conjugacy Problem in Subgroups of Homeomorphism Groups of$\mathbb{R}$ and S^1, in preparation (2009).
[5] M.G. Brin. Higher dimenstional Thompson groups, *Geom. Dedicata*, **108** (2004) 163-192.
[6] J.W. Cannon, W.J. Floyd and W.R. Parry, Introductory notes on Richard Thompson's groups, *Enseign. Math.*, (2) **42**(3-4) (1996) 215-256.
[7] P. M. Cohn, "Universal Algebra". Mathematics and its Applications, 6, D. Reidel Pub. Company, (1981).
[8] V. Guba and M. Sapir, Diagram groups, *Mem. Amer. Math. Soc.*, **130** (620) (1997) viii+117.
[9] G. Higman, Finitely presented infinite simple groups, *Notes on Pure Mathematics*, Vol. 8 (1974).
[10] B. Jónsson and A. Tarski, On two properties of free algebras, *Math. Scand.*, **9** (1961) 95-101.
[11] M. Kassabov and F. Matucci, The Simultaneous Conjugacy Problem in Groups of Piecewise Linear Functions, http://arxiv.org/abs/math/0607167 (2006).
[12] E.I. Khukhro, p-Automorphisms of Finite p-Groups, *London Math. Soc. Lecture note Ser.* **246** (Cambridge University Press 1997).

[13] Lipschutz, S., (1966) Generalisation of Dehn's result on the conjugacy problem, *Proc. Amer. Math. Soc.*, **17** 759–762.

[14] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, second edition, (Dover Publications, 2004).

[15] F. Matucci, Algorithms and Classifications in Groups of Piecewise Linear Homeomorphisms, PhD thesis, Cornell University (2008).

[16] Miller III, C.F., On group-theoretic decision problems and their classification, *Annals Of Mathematics Studies* 68, (Princeton University Press 1971).

[17] Pride, S. J., (2008) On the residual finiteness and other properties of (relative) one-relator groups, *Proc. Amer. Math. Soc.* **136** (2) 377–386.

[18] C. Röver, Subgroups of Finitley Presented Simple Groups. PhD thesis, University of Oxford, Pembroke College, Trinity Term (1999).

[19] O.P. Salazar-Diaz. Thompson's group V from a dynamical viewpoint, *Internat. J. Algebra Comput.*, **20**(1) (2010) 39-70.