NEW EXAMPLES OF BEAUVILLE SURFACES.

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Abstract. We find new mixed and unmixed Beauville structures for groups constructed as 2-quotients, *i.e.* quotients of 2-power order, of the fundamental group of a certain simplicial complex.

1. Introduction

In recent years a number of people have been interested in the study of Beauville surfaces and finite groups (see, e.g., [FJ09], [GP09], [BCP10], [Gar10], [GLL10] and[FGDJZ]).

A Beauville surface S is an infinitesimally rigid (i.e., does not admit non-trivial deformations) complex algebraic surface, which is *isogenous to a higher product*. This means that S is of the form $(C_1 \times C_2)/G$, where C_1 and C_2 are non-singular projective curves of genera $g(C_i) \geq 2$, and G is a finite group acting freely on the product of curves by holomorphic transformations. Let H denote the subgroup of G consisting of the elements of G which preserve each of the curves C_i. The presentation $S \cong (C_1 \times C_2)/G$ is called *minimal* if H acts effectively on each curve C_i. Every Beauville surface S has a unique minimal presentation $(C_1 \times C_2)/G$. Moreover, the corresponding quotients C_i/H are isomorphic to $\mathbb{P}^1$, and the projections $C_i \to \mathbb{P}^1$ are branched coverings, ramified over three points. These properties are equivalent to the rigidity of Beauville surfaces, which means that Beauville surfaces represent isolated points in the moduli space of surfaces of general type.

A Beauville surface $(C_1 \times C_2)/G$ is said to be of mixed or unmixed type, according to whether $[G : H] = 2$ (i.e., G contains elements interchanging the curves C_1 and C_2) or $H = G$, and the group G is said to admit a *mixed* or *unmixed Beauville structure*, respectively. In the mixed case, the curves C_1 and C_2 are necessarily biholomorphic.

Beauville's original example [Bea78] had two curves $C_1 = C_2$, given by the Fermat curve $x^5 + y^5 + z^5 = 0$, and G the group $(\mathbb{Z}/5\mathbb{Z})^2$ acting on $C_1 \times C_2$ by the rule

$$(a, b) \cdot ([x : y : z], [u : v : w]) = ([\xi^a x : \xi^b y : z], [\xi^{a+3b} u : \xi^{2a+4b} v : w]),$$

where $\xi = e^{\frac{2\pi i}{5}}$ and $a, b \in \mathbb{Z}/5\mathbb{Z}$. Then S is a Beauville surface of unmixed type with $g(C_1) = g(C_2) = 6$.

Most of what is known about Beauville surfaces is due to the work of Catanese in [Cat00] and the joint work of Bauer, Catanese and Grunewald in [BCG05, BCG06] and [BCG08]. However, very few examples of Beauville surfaces of mixed type are known. In this paper

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we give a family of them associated to 2-groups, which arise as quotients of a particular infinite group Γ related to a Euclidean building of type $\widetilde{A}_2$.

More precisely, let

$$\Gamma = \langle x_0, ..., x_6 \mid x_i x_{i+1} x_{i+3} \text{ for } i = 0, ..., 6 \rangle,$$

and H be the index 2 subgroup generated by x_0, x_1. Moreover, let $G_{2,k}$ and $H_{2,k}$ denote the maximal 2-quotients of class k of Γ and H, respectively (for more details see Sections 3 and 4 below). We find that $H_{2,k}$ has an *unmixed Beauville structure* for every $3 \leq k \leq 63$ which is not a power of 2 (see Theorem 4.1), and that $G_{2,k}$ admits a *mixed Beauville structure* for every $3 \leq k \leq 10$ which is not a power of 2 (see Theorem 4.2). We conjecture that both results hold generally for all classes $k \geq 3$ which are not powers of 2.

2. Beauville surfaces and their groups

2.1. **Beauville surfaces.** Inspired by a construction by Beauville, Catanese defined in [Cat00] a *Beauville surface* to be a rigid (i.e., it admits no nontrivial deformation) compact complex surface which is isogeneous to a higher product, i.e., it admits an unramified covering which is isomorphic to a product of curves of genera ≥ 2. It was shown in [Cat00] that every such surface has a unique minimal realisation $S := (C_1 \times C_2)/G$, that is the genera of the curves C_1 and C_2 are minimal. Moreover, the action of G on $C_1 \times C_2$ is free and respects the product decomposition, i.e., the elements of G either interchange the factors or act independently on both factors. This allows us to distinguish two types of Beauville surfaces:

Definition 2.1. Let S be a surface isogenous to a product with minimal realisation $S \cong (C_1 \times C_2)/G$. We say that S is a *mixed case* if the action of G exchanges the two factors (and then C_1 and C_2 are isomorphic) and an *unmixed case* if G acts via a diagonal action.

The definition of Beauville surfaces allows a purely group theoretic intrinsic description of all groups producing them. This collection of group theoretical properties is called a *Beauville structure*, and it is discussed, e.g., in [BCG05] how a group G with a Beauville structure gives rise to a corresponding Beauville surface with minimal realisation $(C_1 \times C_2)/G$. As in the case of Beauville surfaces, there is a mixed and unmixed Beauville structure.

Bauer, Catanese and Grunewald [BCG08] used this group theoretical description to classify all smooth complex projective surfaces S isogenous to a product, which are regular, i.e., $q(S) = h^{1,0}(S) = 0$, and have vanishing geometric genus $p_g(S) = h^{2,0}(S) = 0$. Since a surface S isogeneous to a higher product is of general type, $p_g(S) = 0$ implies $q = 0$, because of $\chi(S) = 1 + p_g(S) - q(S) \geq 1$. Furthermore, Beauville surfaces with $p_g(S) \geq 1$ have also vanishing irregularity $q(S) = 0$, since $q(S) = g(C_1/H) + g(C_2/H)$ (see [Ser96, Prop. 2.2]) and $C_1/H \cong C_2/H \cong \mathbb{P}^1$.

It turns out that one of the groups in the classification of [BCG08], which they call $G(256, 2)$ and which has a mixed Beauville structure, coincides with the maximal 2-quotient of class 3 of a fundamental group Γ of a certain simplicial complex, which we define in

Section 3 below. More details about mixed Beauville structures for this group are discussed in Section 3.2.

As mentioned in the introduction, not many Beauville surfaces of mixed type are known. It was shown in [BCG05, Theorem 4.3], that if G admits a mixed Beauville structure, then the index 2 subgroup H must be non abelian, and it was mentioned in [BCG08, Remark 4.2] that no group of order < 256 admits a mixed Beauville structure. Moreover, the classification in [BCG08] implies that there are only two groups occuring in minimal realisations $S \cong (C_1 \times C_2)/G$ with $p_g(S) = q(S) = 0$ admitting mixed Beauville structures, and both of them are of order 256.

Our main aim in the subsequent sections below is to show that not only the group $G(256, 2)$ in [BCG08], but also many other maximal 2-quotients of our group Γ produce Beauville surfaces of mixed type. Since their orders are higher 2-powers than $256 = 2^8$, these other surfaces must necessarily have $p_g(S) \geq 1$.

2.2. **Group theoretical structures.** Following [BCG08] closely, we introduce group theoretic notions which lead to the definition of Beauville structures.

Let G be a finite group and r an integer with $r \geq 2$. An r-tuple $T = [g_1, ..., g_r]$ of elements of G is called a *spherical system of generators*, if $g_1, ..., g_r$ generate G and we additionally have $g_1...g_r = 1$.

For an r-tuple $T = [g_1, ..., g_r]$ of elements of G and $g \in G$, we set

$$gTg^{-1} := [gg_1g^{-1}, ..., gg_rg^{-1}].$$

If $A = [m_1, ..., m_r]$ is an r-tuple of natural numbers with $2 \leq m_1 \leq ... \leq m_r$, then the spherical system of generators $T = [g_1, ..., g_r]$ is said to be *of type* A, if there is a permutation $\tau \in Sym(r)$ such that we have

$$ord(g_1) = m_{\tau(1)}, ord(g_2) = m_{\tau(2)}, \cdots, ord(g_r) = m_{\tau(r)}.$$

(Here $ord(g)$ is the order of the element $g \in G$.)

For a spherical system of generators $T = [g_1, ..., g_r]$ of G, we define

$$\Sigma(T) = \Sigma([g_1, ..., g_r]) := \bigcup_{g \in G} \bigcup_{j=0}^{\infty} \bigcup_{i=1}^{r} \{g \cdot g_i^j \cdot g^{-1}\} \tag{1}$$

to be the union of all conjugates of the cyclic subgroups generated by the elements $g_1, ..., g_r$. A pair of spherical systems of generators (T_1, T_2) of G is called *disjoint* if

$$\Sigma(T_1) \cap \Sigma(T_2) = \{1\}.$$

Next, we introduce unmixed and mixed ramification structures.

Definition 2.2. Let $A_1 = [m_{(1,1)}, .., m_{(1,r)}]$ and $A_2 = [m_{(2,1)}, .., m_{(2,s)}]$ be tuples of natural numbers with $2 \leq m_{(1,1)} \leq ... \leq m_{(1,r)}$ and $2 \leq m_{(2,1)} \leq ... \leq m_{(2,s)}$. An *unmixed ramification structure of type* (A_1, A_2) *for* G is a disjoint pair (T_1, T_2) of spherical systems of generators, such that T_1 has type A_1 and T_2 has type A_2.

The disjointness of the pair (T_1, T_2) in the definition of an unmixed ramification structure guarantees that G acts freely on the product $C_{T_1} \times C_{T_2}$ of the associated algebraic curves (see Section 4.3 and the references therein).

Definition 2.3. Let $A = [m_1, ..., m_r]$ be an r-tuple of natural numbers with $2 \leq m_1 \leq ... \leq m_r$. A *mixed ramification structure of type A for G* is a pair (H, T) where H is a subgroup of index 2 in G and $T = [g_1, ...g_r]$ is an r-tuple of elements of G such that the following hold:

- T is a spherical system of generators of H of type A,
- for every $g \in G \setminus H$, the spherical systems T and gTg^{-1} are disjoint,
- for every $g \in G \setminus H$ we have $g^2 \notin \Sigma(T)$.

Definition 2.4. An *unmixed Beauville structure* is an unmixed ramification structure with two spherical systems (T_1, T_2) of generators of length 3, i.e., $r = 3$ and $s = 3$. A *mixed Beauville structure* is a mixed ramification structure with a spherical system T of generators of length 3, i.e., $r = 3$.

Remark 2.5. We note that if G has a mixed Beauville structure (H, T) of type A, then the index 2 subgroup H has an unmixed Beauville structure of type (A, A), by choosing the pair (T, gTg^{-1}) for an arbitrary $g \in G \setminus H$ and using the fact that H is normal in G. Moreover, there are corresponding unmixed and mixed Beauville surfaces $S_H = (C_1 \times C_2)/H$ and $S_G = (C_1 \times C_2)/G$, so that S_H is a 2-fold covering of S_G.

As [GP09] states, the question of which finite groups admit an unmixed Beauville structure is deeply related to the question of which finite groups are quotients of certain triangle groups; a survey about this is given in [Con02].

3. The group Γ and a 2-quotient with a mixed Beauville structure

3.1. **The fundamental group Γ.** We consider a simplicial complex $\mathcal{K}$, built up by 7 triangles, and defined by the labeling scheme given in Figure 3.1 below. Note that all vertices of the triangles represent the same point in $\mathcal{K}$. Let $\Gamma = \pi_1(\mathcal{K})$ be the fundamental group of the complex $\mathcal{K}$. Realising all triangles geometrically by equilateral Euclidean triangles, we can view the universal covering of $\mathcal{K}$ as a thick Euclidean building of type $\widetilde{A}_2$, with Γ being isomorphic to the group of covering transformations. The group Γ belongs to a family of groups introduced in [CMSZ93], and is obviously presented by

$$\Gamma = \langle x_0, ..., x_6 \mid x_i x_{i+1} x_{i+3} \text{ for } i = 0, ..., 6 \rangle,$$

where i, $i+1$ and $i+3$ are taken modulo 7.

In [PV08], we considered the subgroup $H \subset \Gamma$, generated by x_0 and x_1, and employed 2-quotients of this subgroup for the explicit construction of expander graph families. We recall the following fact from [PV08]:

Proposition 3.1. *The group Γ is generated by x_0, x_1, x_2, and the subgroup H, generated by x_0, x_1 is an index two normal subgroup of Γ. Moreover, H has the presentation*

$$H = \langle x_0, x_1 \mid r_1, r_2, r_3 \rangle,$$

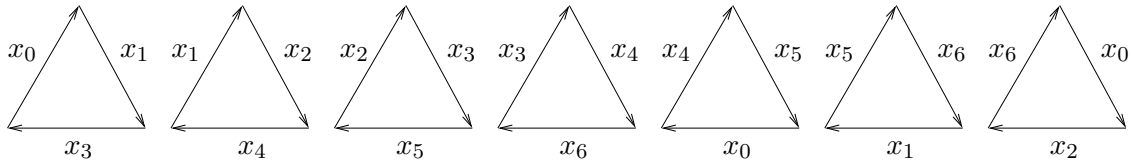

FIGURE 3.1. Labeling scheme for a simplicial complex

with

$$\begin{aligned} r_1 &= x_1x_0x_1x_0x_1x_0x_1^{-3}x_0^{-3}, \\ r_2 &= x_1x_0^{-1}x_1^{-1}x_0^{-3}x_1^2x_0^{-1}x_1x_0x_1, \\ r_3 &= x_1^3x_0^{-1}x_1x_0x_1x_0^2x_1^2x_0x_1x_0. \end{aligned}$$

3.2. **A group with a mixed Beauville structure.** Let $G_{2,3}$ denote the maximal 2-quotient of class 3 of Γ. Similarly, let $H_{2,3}$ denote the maximal 2-quotient of class 3 of the subgroup H. $H_{2,3}$ is an index 2 subgroup of $G_{2,3}$. ($G_{2,3}$ and $H_{2,3}$ coincide with the groups SmallGroup$(256, 3679)$ and SmallGroup$(128, 36)$ in MAGMA Small Groups notation.) To simplify notation, we denote the images of x_0 and x_1 in $G_{2,3}$, again, by x_0 and x_1.

The quotient $G_{2,3}$ is of order $256 = 2^8$, and coincides with the group (they call) $G(256, 2)$ in Bauer, Catanese and Grunewald [BCG08]. They constructed in [BCG08, Section 6.6] a mixed Beauville structure of type $[4, 4, 4]$ for this group.

In our notation, we establish a mixed Beauville structure of type $A = [4, 4, 4]$ for the group $G_{2,3}$, by choosing $(H_{2,3}, T_1)$, where T_1 is the spherical system of generators $[x_0, x_1, x]$ with $x = x_1^{-1}x_0^{-1}$. A second mixed Beauville structure of type A is given by the pair $(H_{2,3}, T_2)$ with $T_2 = [y_0, y_1, y]$, where $y_0 = x_0x_1^{-1}$ and $y_1 = x_1x_0x_1$ is another set of generators of $H_{2,3}$ and $y = y_1^{-1}y_0^{-1}$. Moreover, the disjoint pair (T_1, T_2) of spherical systems of generators is an unmixed Beauville structure of type (A, A) for the group $H_{2,3}$.

These facts were confirmed by MAGMA calculations, and lead to the study of Beauville structures for other maximal 2-quotients of the groups Γ and H.

4. Beauville structures for maximal 2-quotients

Before we present our results and conjectures, we fix some notation. We denote the maximal 2-quotient of class k of Γ by $G_{2,k}$, and the maximal 2-quotient of class k of H by $H_{2,k}$. For simplicity, we denote the images of x_0 and x_1 in $G_{2,k}$, again, by x_0 and x_1. We define $y_0 = x_0x_1^{-1}$, $y_1 = x_1x_0x_1$, and $x = x_1^{-1}x_0^{-1}$, $y = y_1^{-1}y_0^{-1}$, as well as the spherical systems of generators $T_1 = [x_0, x_1, x]$ and $T_2 = [y_0, y_1, y]$ for the groups $H_{2,k}$. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the largest integer smaller or equal to x. For an integer k, let $k \bmod 3 \in \{0, 1, 2\}$ be the remainder under integer division by 3.

4.1. **Which groups $H_{2,k}$ admit unmixed Beauville structures (T_1, T_2)?** Our MAGMA calculations show the following result for the maximal 2-quotients $H_{2,k}$ of class $k \leq 64$:

Theorem 4.1. *Let $3 \leq k \leq 63$, $r = \lfloor \log_2 k \rfloor + 1$ and $A = [2^r, 2^r, 2^r]$. If k is not a power of 2, then (T_1, T_2) is an* unmixed Beauville structure *of type (A, A) for the group $H_{2,k}$.*

In the case that k is a power of 2, i.e., $k = 2^s$ for $s = 2, 3, \ldots, 6$, we found that $x_0^k = y_0^k$ and $x_1^k = y_1^k$, which means that (T_1, T_2) is not an unmixed Beauville structure of $H_{2,k}$. Moreover, the conjugacy classes of x_0^k and x_1^k in $G_{2,k}$ are trivial. Therefore, none of the pairs (T_1, gT_2g^{-1}) with $g \in G_{2,k}$ can be an unmixed Beauville structure of $H_{2,k}$.

We conjecture that all of the above results hold not only for $3 \leq k \leq 64$, but for all integers $k \geq 3$.

4.2. **Which groups $G_{2,k}$ have mixed Beauville structures?** Since MAGMA calculations for mixed structures are far more intensive than for a particular unmixed structure, we confined our MAGMA calculations to all 2-quotients of classes up to $k \leq 10$ and obtained the following result:

Theorem 4.2. *Let $3 \leq k \leq 10$ and $r = \lfloor \log_2 k \rfloor + 1$. If k is not a power of 2, then $(H_{2,k}, T_1)$ is a* mixed Beauville structure *of type $[2^r, 2^r, 2^r]$ for the group $G_{2,k}$.*

Again, we conjecture that this theorem holds for all integers $k \geq 3$ which are not powers of 2.

4.3. **What Beauville surfaces do these groups correspond to?** It is described, e.g., in [BCG05] (or in [BCG06]) how to construct, for a given finite group G with a Beauville structure, a corresponding Beauville surface with minimal realisation $(C_1 \times C_2)/G$. We only sketch this construction. Let $T = [g_1, g_2, g_3]$ be a spherical system of generators of G of type $A = [a_1, a_2, a_3]$. Let $B = \{-1, 0, 1\} \subset \mathbb{P}$, and we fix the point $\infty \in \mathbb{P}$. Employing Riemann's existence theorem, this data is used to construct an explicit surjective homomorphism $\pi_1(\mathbb{P}^1 \backslash B, \infty) \to G$ with $g_1 \mapsto \alpha, g_2 \mapsto \beta, g_3 \mapsto \gamma$, where α, β, γ are particularly chosen generators of $\pi_1(\mathbb{P}^1 \backslash B, \infty)$ satisfying $\alpha\beta\gamma = 1$, and a corresponding Galois covering $C_T \to \mathbb{P}^1$ with group G, ramified in the three points $-1, 0, 1 \in \mathbb{P}$, with ramification indices equal to the orders a_1, a_2 and a_3, respectively. The Riemann-Hurwitz formula yields for the genus $g(C_T)$ of the curve C_T,

$$2g(C_T) - 2 = |G|(1 - \mu(A)), \tag{2}$$

where $\mu(A) = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$. Assuming that G has an unmixed Beauville structure (T_1, T_2) of type (A_1, A_2), the corresponding Beauville surface is constructed as $S_G = (C_{T_1} \times C_{T_2})/G$. We have for the topological Euler number $e(S_G)$, by the Theorem of Zeuthen-Segre,

$$e(S_G) = 4\frac{(g(C_{T_1}) - 1)(g(C_{T_2} - 1)}{|G|},$$

as well as the relations (see [Cat00, Theorem 3.4]),

$$\chi(S_G) = \frac{e(S_G)}{4} = \frac{K_{S_G}^2}{8},$$

where $K_{S_G}^2$ denotes the self intersection number of the canonical divisor.

Finally, since S_G is a surface of general type and $q(S_G) = 0$, we have

$$\chi(S_G) = 1 + p_g(S_G) = \frac{|G|}{4}(1 - \mu(A_1))(1 - \mu(A_2)). \tag{3}$$

These relations allow us to calculate all main invariants of Beauville surfaces corresponding to the groups $H_{2,k}$ and $G_{2,k}$, for which our MAGMA calculations showed the existence of unmixed and mixed Beauville structures:

(1) Let $H_{2,k}$ have an unmixed Beauville structure of type (A, A) with $A = [2^r, 2^r, 2^r]$ and $r = \lfloor \log_2 k \rfloor + 1$. Let $S_{H_{2,k}} = (C_1 \times C_2)/H_{2,k}$ be the minimal realisation of an associated Beauville surface. Since $H_{2,k}$ is a group of order $2^{8\lfloor k/3 \rfloor + 3\cdot(k \bmod 3) - 1}$, we obtain from (2),

$$g(C_1) - 1 = g(C_2) - 1 = 2^{8\lfloor k/3 \rfloor + 3\cdot(k \bmod 3) - r - 2}(2^r - 3),$$

and from (3),

$$1 + p_g(S_{H_{2,k}}) = \chi(S_{H_{2,k}}) = 2^{8\lfloor k/3 \rfloor + 3\cdot(k \bmod 3) - 2r - 3}(2^r - 3)^2.$$

(2) Let $G_{2,k}$ have a mixed Beauville structure of type $A = [2^r, 2^r, 2^r]$ with $r = \lfloor \log_2 k \rfloor + 1$. Then $H_{2,k}$ admits unmixed Beauville structures of type (A, A) by Remark 2.5, and we have a 2-fold covering $S_{H_{2,k}} \to S_{G_{2,k}}$ of associated Beauville surfaces. This implies

$$1 + p_g(S_{G_{2,k}}) = \chi(S_{G_{2,k}}) = \frac{1}{2}\chi(S_{H_{2,k}}) = 2^{8\lfloor k/3 \rfloor + 3\cdot(k \bmod 3) - 2r - 4}(2^r - 3)^2.$$

In the particular case $G_{2,3} \cong G(256, 2)$, we recover the results $g(C_1) = g(C_2) = 17$ and vanishing geometric genus $p_g(S_{G_{2,3}}) = 0$, in accordance with [BCG08].

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