

# SYMBOL DECOMPOSITIONS FOR ELEMENTS OF THOMPSON'S GROUP $F$

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**ABSTRACT.** We view Thompson's group  $F$  as a subgroup  $F_{2,1}$  of the automorphism group  $G_{2,1}$  of the free algebra defined by Graham Higman in order to represent any finitely generated subgroup of the group  $F_{2,1}$  by a finite set of directed abstract graphs.

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## 1. INTRODUCTION

The world of Thompson's group  $F$  is vast and there are many ways to study the group. Such as from the point of view of dynamics [3, 10], combinatorics [6] and even diagrams [8, 2]. Many more references could be given to these approaches.

Here, we will take advantage of the work of Graham Higman in [9] which uses an algebraic description of a larger group  $G_{2,1}$  (also known as Thompson's group  $V$ ) of which Thompson's group  $F$  is a subgroup. We will call this subgroup  $F_{2,1}$  (in line with [4]). In the proceeding sections we will present some properties that the group  $G_{2,1}$  possess and demonstrate an algorithm (first described in [9]) that decomposes elements of the group  $G_{2,1}$ , which also can be used for elements of the subgroup  $F_{2,1}$ . This decomposition is very different to the minimal length representatives of elements achieved in [7] *i.e* there is no uniqueness (of a decomposition) or minimality (in terms of word length) in a single decomposition for an element using the algorithm in this note.

**1.1. The Higman-Thompson group  $G_{2,1}$ .** We recall the definition of the free algebra  $V_{2,1}$  defined by the set  $S$  (described below) and the operations on the set  $S$ ; two unary operations  $\alpha_1, \alpha_2$  and one binary operation  $\lambda$ .

**Definition 1.1.** The set  $S$  of strings in *standard form* is defined as the smallest subset of  $\{x, \alpha_1, \alpha_2, \lambda\}^*$  as follows (where  $A = \{\alpha_1, \alpha_2\}$ ):

- $x \in S$ ,
- $x\Gamma \in S$  for  $\Gamma \in \langle A \rangle$ ,
- if  $w_1, w_2 \in S$  and there is no  $u \in S$  with  $w_i = u\alpha_i$  for  $i = 1, 2$  then  $w_1w_2\lambda \in S$ .

We define unary operations  $\alpha_1, \alpha_2$  and a binary operation  $\lambda$  on the set  $S$  of standard forms by:

- $(x\Gamma)\alpha_i = x\Gamma\alpha_i$ ,

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- $(w_1, w_2)\lambda = w_1w_2\lambda$  unless there exists  $u \in S$  with  $w_i = u\alpha_i$  for  $i = 1, 2$  then  $(u\alpha_1, u\alpha_2)\lambda = u$ ,
- $((w_1, w_2)\lambda)\alpha_i = w_i$ .

*Remark 1.2.* We write  $\langle A \rangle$  to denote the free monoid generated by  $\alpha_1, \alpha_2$ . The singleton set  $\{x\}$  will be denoted by  $\mathbf{x}$ .

For a subset  $X$  of the algebra  $V_{2,1}$ , we shall write  $X\langle A \rangle$  for the  $A$ -subalgebra of  $V_{2,1}$  generated by elements of  $X$  under the application of the operations  $\alpha_i$  for  $i = 1, 2$ , and  $X\langle \lambda \rangle$  for the  $\lambda$ -subalgebra generated by elements of  $X$  under the application of the operation  $\lambda$ . It is possible to see that  $V_{2,1} = \mathbf{x}\langle A \rangle\langle \lambda \rangle$ .

**Lemma 1.3.** [9, Lemma 2.3 (i),(ii)] *Suppose that  $X$  is a free generating set for  $V_{2,1}$  and that  $y, y_1, y_2$  are distinct elements of  $X$ . Then*

(1)

$$(X \setminus \{y\}) \cup \{z \mid z = (y)\alpha_i, i = 1, 2\}$$

*is also a free generating set. Note that  $(y)\alpha_i = y\alpha_i$  for  $i = 1, 2$ . We call this new generating set a single expansion of  $X$ ; a finite composition of  $d$  single expansions is called a  $d$ -fold expansion of  $X$ .*

(2)

$$(X \setminus \{y_1, y_2\}) \cup \{(y_1, y_2)\lambda\}$$

*is also a free generating set. This is called a single contraction of  $X$ ; a finite composition of single contractions is called a contraction.*

Higman proved that any expansion of  $X$  freely generates  $V_{2,1}$  and that any single contraction of  $X$  freely generates  $V_{2,1}$ . This implies that  $\mathbf{x}$  freely generates the algebra  $V_{2,1}$ .

**Definition 1.4.** We define a subset of  $\mathbf{x}\langle A \rangle$  to be called a *basis* of  $\mathbf{x}\langle A \rangle$  if it is a  $d$ -fold expansion of  $\mathbf{x}$  for some  $d \in \mathbb{N}$ . We then say that this basis is a finite free basis of  $\mathbf{x}\langle A \rangle$  of size  $d$ .

Let  $G_{2,1}$  be the automorphism group of the free algebra  $V_{2,1}$ , which is freely generated by  $\mathbf{x}$ . Let  $Y, Z$  be finite free bases of  $\mathbf{x}\langle A \rangle$  of size  $d$ . Then any map  $\psi : Y \rightarrow Z$  induces an automorphism of  $V_{2,1}$ . We have the following lemma from [9].

**Lemma 1.5.** [9, Lemma 4.1] *If  $\{\psi_1, \dots, \psi_k\}$  is a finite subset of  $G_{2,1}$  then there is a unique minimal basis  $Y$  of  $\mathbf{x}\langle A \rangle$  such that  $Y\psi_i \subseteq \mathbf{x}\langle A \rangle$ , for  $i = 1, \dots, k$ . Any other basis of  $\mathbf{x}\langle A \rangle$  with this property is an expansion of  $Y$ .*

## 2. GRAHAM HIGMAN'S SYMBOLS

We introduce the basic concepts given in [9, Section 4].

**2.1. Depth of an element or finite subset of elements of  $G_{2,1}$ .** We know by Lemma 1.5 that for a finite subset of elements  $\{\psi_1, \dots, \psi_s\}$  of  $G_{2,1}$  there exists a unique minimal expansion  $Y$  of  $x$  such that  $Y\psi_i \subseteq x\langle A \rangle$  for  $i = 1, \dots, s$ .

**Definition 2.1.** [9] If  $Y$  is a  $d$ -fold expansion of  $x$ , so that  $|Y| = 1 + d$ , we say that the set  $H = \{\psi_1, \dots, \psi_s\}$  has depth  $d$ .

**Example 2.2.** Let  $\psi$  be defined by the bases

$$Y = \{x\alpha_1\alpha_1, x\alpha_1\alpha_2, x\alpha_2\}$$

and

$$Z = \{x\alpha_2\alpha_2, x\alpha_2\alpha_1, x\alpha_1\}$$

and the equation

$$y_i\psi = z_i,$$

with  $i = 1, 2, 3$  with the natural ordering on the elements of  $Y$  and  $Z$ . Then  $\psi$  has depth  $|Y| - 1 = 3 - 1 = 2$ .

**2.2. Basics on Symbols.** We now describe the tools developed by Higman in [9], which were used to give finite presentability and a solution to the word problem for the groups  $G_{n,r}$ .

**Definition 2.3.** [9] A symbol for  $\psi \in G_{2,1}$  is an expression,

$$\begin{pmatrix} y_1 & \dots & y_N \\ z_1 & \dots & z_N \end{pmatrix},$$

where  $Y = \{y_1, \dots, y_N\}$  and  $Z = \{z_1, \dots, z_N\}$  are expansions of  $x$  such that  $y_i\psi = z_i$  for  $i = 1, \dots, N$ .

It can be shown, [9, Section 4], that every element of  $G_{2,1}$  has a symbol. If  $N = 1 + d$  then it is said that  $\psi$  has depth at most  $d$ . If  $\psi$  has depth less than  $d$ , then since any expansion whose image under  $\psi$  in  $x\langle A \rangle$  is an expansion of a minimal one, the columns of the symbol can be reordered so that it has the form,

$$\begin{pmatrix} y_1\alpha_1 & y_1\alpha_2 & y_2 & \dots & y_N \\ z_1\alpha_1 & z_1\alpha_2 & z_2 & \dots & z_N \end{pmatrix}.$$

Similarly, a symbol for a relation  $\psi_1 \dots \psi_s = \psi$  is given by,

$$\begin{pmatrix} y_{11} & \dots & y_{1N} \\ \vdots & & \vdots \\ z_{s+1,1} & \dots & z_{s+1,N} \end{pmatrix},$$

where for  $i = 1, \dots, s$ ,

$$\begin{pmatrix} y_{i,1} & \dots & y_{i,N} \\ z_{i+1,1} & \dots & z_{i+1,N} \end{pmatrix},$$

is a symbol for  $\psi_i$  and hence

$$\begin{pmatrix} y_{1,1} & \dots & y_{1,N} \\ z_{s+1,1} & \dots & z_{s+1,N} \end{pmatrix},$$

is a symbol for  $\psi$ . If  $N = 1 + d$ , then the relation has depth at most  $d$  and depth less than  $d$  if and only if (upon rearrangement of the columns) the symbol has the form,

$$\begin{pmatrix} u_1\alpha_1 & u_1\alpha_2 & y_{1,1} & \cdots & y_{1,N} \\ u_2\alpha_1 & u_2\alpha_2 & y_{2,1} & \cdots & y_{2,N} \\ \vdots & \vdots & \vdots & & \vdots \\ u_s\alpha_1 & u_s\alpha_2 & z_{s,1} & \cdots & z_{s,N} \\ u_{s+1}\alpha_1 & u_{s+1}\alpha_2 & z_{s+1,1} & \cdots & z_{s+1,N} \end{pmatrix}.$$

Symbols make the word problem in  $G_{2,1}$  very easy. As every element of  $G_{2,1}$  has a symbol and we can always find a unique minimal common basis  $Y$  for a subset of elements, we can just check directly to see if in a symbol for  $\psi_1 \dots \psi_s$  we have  $y_{1,j} = z_{s+1,j}$  for all  $j = 1, \dots, N$ .

**Example 2.4.** From Example 2.2, we had the element  $\psi$ . A symbol for  $\psi$  would be,

$$\begin{pmatrix} x\alpha_1\alpha_1 & x\alpha_1\alpha_2 & x\alpha_2 \\ x\alpha_2\alpha_2 & x\alpha_2\alpha_1 & x\alpha_1 \end{pmatrix}.$$

If we were to put  $\psi$  in quasi-normal form with respect to a basis  $Y$  (see [9, Section 9] for the definition or [1, Definition 21]), the basis would be  $Y = \{x\alpha_1\alpha_1, x\alpha_1\alpha_2, x\alpha_2\alpha_1, x\alpha_2\alpha_2\}$ . This would give a symbol,

$$\begin{pmatrix} x\alpha_1\alpha_1 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2\alpha_2 \\ x\alpha_2\alpha_2 & x\alpha_2\alpha_1 & x\alpha_1\alpha_1 & x\alpha_1\alpha_2 \end{pmatrix}.$$

This symbol has depth 3, but the element  $\psi$  has depth 2 (notice the last two columns).

*Remark 2.5.* Graham Higman's symbols were generalized by Scott in [12] for use with a larger group  $\mathcal{G}_{n,r}$  which contains  $G_{n,r}$  and used by Röver in [11].

### 3. THOMPSON'S GROUP $F$

The standard introduction to the properties and the presentations of Thompson's group  $F$  is [5]. The thesis [2] is also essential reading for a more recent account of the known results. However, here we will view Thompson's Group  $F$  as a subgroup  $F_{2,1}$  of the automorphism group  $G_{2,1}$  of the free algebra  $V_{2,1}$ . This subgroup was originally viewed and defined in this way by Brown in [4].

The map  $\psi$  which takes a basis  $Y$  to a basis  $Z$  both of size  $n$ , is an element of  $F_{2,1}$  if  $\psi$  preserves the lexicographic order of the elements of the basis  $Z$ .

**Definition 3.1.** The lexicographic order on the elements of  $\mathbf{x}\langle A \rangle$  is given in the following way: for  $x\Gamma$  and  $x\Delta$  (with  $\Gamma, \Delta \in \langle A \rangle$ ) we say  $x\Gamma <_{lex} x\Delta$  if  $\Gamma <_{lex} \Delta$  where we define  $\alpha_1 <_{lex} \alpha_2$ .

**Definition 3.2.** [4] Let  $F_{2,1}$  be the group of all elements  $\psi$  of  $G_{2,1}$  of the form

$$\psi : Y \rightarrow Z$$

such that with  $y_1 <_{lex} y_2 <_{lex} \dots <_{lex} y_n$  and  $z_1 <_{lex} z_2 <_{lex} \dots <_{lex} z_n$  we have  $y_i\psi = z_i$  for  $i = 1, \dots, n$ .

We have generators for  $F$  represented by the famous tree pairs  $x_0$  and  $x_1$  (see [5] for more information),

$$\begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \end{array} \longrightarrow \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \end{array} \quad \text{and} \quad \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad \diagdown \quad \diagup \\ \quad 2 \quad 3 \quad 4 \end{array} \longrightarrow \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array}$$

The generators  $x_0, x_1$  in  $F_{2,1}$  are given by the symbols

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2 & x\alpha_2 \\ x\alpha_1 & x\alpha_2\alpha_1 & x\alpha_2^2 \end{pmatrix},$$

and

$$\begin{pmatrix} x\alpha_1 & x\alpha_2\alpha_1^2 & x\alpha_2\alpha_1\alpha_2 & x\alpha_2^2 \\ x\alpha_1 & x\alpha_2\alpha_1 & x\alpha_2^2\alpha_1 & x\alpha_2^3 \end{pmatrix}.$$

An important part of the proof in [9] of finite generation of the group  $G_{2,1}$  is the formation of the lemma given below.

**Lemma 3.3.** [9, Lemma 4.2] *If  $d \geq 4$ , an element  $\psi$  of  $G_{n,r}$  of depth  $d$  can be expressed as a product of elements of depth less than  $d$ .*

We will now show that the same result holds for  $F_{2,1}$ , the proof uses the same approach and tools as the proof of Lemma 3.3 given by Higman in [9]. In fact, a proof for  $F_{n,r}$   $n \geq 2, r \geq 1$  can be found in [12, Section 4].

**Corollary 3.4.** *If  $d \geq 4$ , an element  $\psi$  of  $F_{2,1}$  of depth  $d$  can be expressed as a product of elements of depth less than  $d$ .*

*Proof.* As depth  $d \geq 4$  we have from the formula from Definition 2.1 (which also apply to elements in  $F_{2,1} \subset G_{2,1}$ ) that  $N = 1 + d \geq 1 + 4 = 5$  and there exists a  $d$ -fold expansion of  $\mathbf{x}$ , for example without loss of generality, of the form

$$(u\alpha_1, u\alpha_2, v\alpha_1, v\alpha_2, w_5, \dots, w_N).$$

We note that there are finitely many possible  $d$ -fold expansions of  $\mathbf{x}$ .

Let

$$\begin{pmatrix} y_1 & \dots & y_N \\ z_1 & \dots & z_N \end{pmatrix}$$

be a symbol for  $\psi$ . As  $d \geq 1$ , the elements of the basis  $Y = \{y_1, \dots, y_N\}$  contain a subset  $\{a\alpha_1, b\alpha_2\}$  and the elements of the basis  $Z = \{z_1, \dots, z_N\}$  contains a subset  $\{b\alpha_1, b\alpha_2\}$ . We now have two cases:

- (1) If no column of the symbol for  $\psi$  contains more than one of the 4 elements  $a\alpha_1, a\alpha_2, b\alpha_1, b\alpha_2$ . Then, without loss of generality on the position of the 4 elements in the columns, we can insert an extra row in the symbol to obtain for example,

$$\begin{pmatrix} a\alpha_1 & a\alpha_2 & y_3 & y_4 & y_5 & \dots & y_N \\ u\alpha_1 & u\alpha_2 & v\alpha_1 & v\alpha_2 & w_5 & \dots & w_N \\ z_1 & z_2 & b\alpha_1 & b\alpha_2 & z_5 & \dots & z_N \end{pmatrix}.$$

This is a symbol for  $\psi = \psi_1\psi_2$ , where the  $\psi_i$  have depth at most  $d - 1$  for  $i = 1, 2$  and are elements of  $F_{2,1}$ .

- (2) If between  $a\alpha_1, a\alpha_2, b\alpha_1, b\alpha_2$  they occupy at most 3 columns, then we can form the symbol, again without loss of generality on the position of the 4 elements, for example

$$\begin{pmatrix} a\alpha_1 & a\alpha_2 & y_3 & \dots & y_{N-2} & y_{N-1} & y_N \\ u\alpha_1 & u\alpha_2 & \dots & & & v\alpha_1 & v\alpha_2 \\ u\alpha_1 & & \dots & u\alpha_2 & \dots & v\alpha_1 & v\alpha_2 \\ b\alpha_1 & & \dots & b\alpha_2 & \dots & z_{N-1} & z_N \end{pmatrix}.$$

This is a symbol for  $\psi = \psi_1\psi_2\psi_3$ , where the  $\psi_i$  have depth at most  $d - 1$  for  $i = 1, 2$  and are elements of  $F_{2,1}$ .

Thus, we can always reduce to decompositions consisting of elements of depth at most 3.  $\square$

*Remark 3.5.* We will refer to the proof of Lemma 3.4 as the algorithm for decomposing elements  $\psi$  of  $F_{2,1}$  of depth  $d \geq 4$  into elements  $\theta_i, \theta_i^{-1}$  of depth at most 3, were all possible choices (cases (1), (2)) are made at each stage to give rise to all decompositions via this method.

This algorithm terminates as each new element created by the process in the decomposition of  $\psi$  is of depth at most  $d - 1$ .

We first notice that such decompositions are not unique. We may have a choice for the quadruple  $a\alpha_1, a\alpha_2, b\alpha_1, b\alpha_2$ .

If there are more than one pair of elements  $\{a\alpha_1, a\alpha_2\}$  and  $\{b\alpha_1, b\alpha_2\}$  in each basis  $Y$  and  $Z$ , then we have a finite number of choices that will give a finite number of decompositions into elements of depth at most 3.

We will, obviously, exclude the addition of an element of depth 0 (the identity element) and consecutive elements  $\varphi_1\varphi_2$ , if  $\varphi_1\varphi_2 = 1$ . Though, obviously the algorithm given in the proof of Corollary 3.4 (and previously the proof of Lemma 3.3 by Higman) would exclude these possibilities.

This now gives an important corollary about the finite generation of  $G_{2,1}$  and now  $F_{2,1}$ .

**Corollary 3.6.** [9, Corollary, page 27]  $G_{2,1}$  is generated by the elements of depth at most 3.

Again, the following corollary is a specific case of [12, Lemma 13].

**Corollary 3.7.**  $F_{2,1}$  is generated by the elements of depth at most 3.

Each element  $\psi$  of  $F_{2,1}$  of depth  $d \geq 4$  can be decomposed into a product of at most  $3^{d-3}$  elements of depth at most 3, using the algorithm given by the proof of Corollary 3.4.

**Definition 3.8.** Let  $\psi$  be an element of  $F_{2,1}$  of depth at most  $d \geq 4$ . We define  $\mathbb{M}(\psi)$  to be the set of all decompositions of the element  $\psi$  into elements of depth at most 3 given by the algorithm in the proof of Corollary 3.4.

An element  $a_j$  of the set  $\mathbb{M}(\psi)$  is a product of elements  $\theta_i, \theta_i^{-1}$ , whose of depths are at most 3. It is clear that the set  $\mathbb{M}(\psi)$  is finite.

Let us now use the information above to decompose elements of  $F_{2,1}$  into products of elements of depth at most 3. However, in the subgroup  $F_{2,1}$  we have a reduction in the number possibilities for elements of depth at most 3. For one,  $F_{2,1}$  has no torsion elements.

**Lemma 3.9.** *There are 16 elements (including inverses and excluding the identity element) of the group  $F_{2,1}$  whose depths are at most 3.*

*Proof.* A calculation on possible elements of depth at most 3 gives rise to the elements  $\theta_1, \dots, \theta_8$  defined in Definition 3.10.  $\square$

**Definition 3.10.** We shall call the elements of  $F_{2,1}$  of depth at most 3,

$$\begin{aligned}
 \theta_1 : & \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \end{array} \longrightarrow \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \end{array}, & \theta_2 : & \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \longrightarrow \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array}, \\
 \theta_3 : & \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \longrightarrow \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array}, & \theta_4 : & \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \longrightarrow \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array}, \\
 \theta_5 : & \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \longrightarrow \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array}, & \theta_6 : & \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \longrightarrow \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array}, \\
 \theta_7 : & \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \longrightarrow \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array}, & \theta_8 : & \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \longrightarrow \begin{array}{c} \diagup \quad \diagdown \\ 1 \quad 2 \quad 3 \quad 4 \end{array}.
 \end{aligned}$$

*Remark 3.11.* We note that  $\theta_1$  is the only element of  $F_{2,1}$  of depth 2.

*Remark 3.12.* As  $\theta_1, \theta_2$  generate  $F_{2,1}$  we find that  $\theta_3 = \theta_1^2 \theta_2^{-1} \theta_1^{-1}$ ,  $\theta_4 = \theta_1^2$ ,  $\theta_5 = \theta_2 \theta_1^{-1}$ ,  $\theta_6 = \theta_1 \theta_2 \theta_1^{-1}$ ,  $\theta_7 = \theta_1 \theta_2$  and  $\theta_8 = \theta_1^2 \theta_2^{-1}$ .

We can then use the techniques for symbol decomposition for elements with depth  $d \geq 4$ , but with the restrictions to the rows that are added. We can only add new rows that give elements of  $F_{2,1}$ . We now demonstrate with an example.

**Example 3.13.** Let  $\psi$  be an element of  $F_{2,1}$  defined by the symbol of depth 5,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2\alpha_1 & x\alpha_1\alpha_2^3 & x\alpha_2\alpha_1 & x\alpha_2^2 \\ x\alpha_1^5 & x\alpha_1^4\alpha_2 & x\alpha_1^3\alpha_2 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix}.$$

We notice that the element  $\psi$  has depth 5 as there are no reductions possible in the symbol.

Let us choose a quadruple of elements from the symbol,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2\alpha_1 & x\alpha_1\alpha_2^3 & x\alpha_2\alpha_1 & x\alpha_2^2 \\ x\alpha_1^5 & x\alpha_1^4\alpha_2 & x\alpha_1^3\alpha_2 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix}.$$

We can then choose a basis for  $x\langle A \rangle$  of size 6 that will give a decomposition into two elements of  $F_{2,1}$  of depth at most 4, say

$$X = \{x\alpha_1^3, x\alpha_1^2\alpha_2, x\alpha_1\alpha_2, x\alpha_2\alpha_1, x\alpha_2^2\alpha_1, x\alpha_2^3\}.$$

We now insert the new row into the symbol for  $\psi$ ,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2\alpha_1 & x\alpha_1\alpha_2^3 & x\alpha_2\alpha_1 & x\alpha_2^2 \\ x\alpha_1^3 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2^2\alpha_1 & x\alpha_2^3 \\ x\alpha_1^5 & x\alpha_1^4\alpha_2 & x\alpha_1^3\alpha_2 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix}.$$

This gives rise to two elements  $\varphi_1$  and  $\varphi_2$ . We notice that the symbol for  $\varphi_2$  actually reduces to

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2^2\alpha_1 & x\alpha_2^3 \\ x\alpha_1^4 & x\alpha_1^3\alpha_2 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix},$$

and then to,

$$\begin{pmatrix} x\alpha_1 & x\alpha_2\alpha_1 & x\alpha_2^2\alpha_1 & x\alpha_2^3 \\ x\alpha_1^3 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix}.$$

So,  $\varphi_2$  is actually the generator  $\theta_4^{-1}$ .

We now turn our attention to the first two rows and recover the element  $\varphi_1$  given by the symbol,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2\alpha_1 & x\alpha_1\alpha_2^3 & x\alpha_2 \\ x\alpha_1^3 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2^2 \end{pmatrix}.$$

We choose a basis of size 5 to be

$$X = \{x\alpha_1^2, x\alpha_1\alpha_2, x\alpha_2\alpha_1^2, x\alpha_2\alpha_1\alpha_2, x\alpha_2^2\},$$

in order to create elements of  $F_{2,1}$  and a quadruple of elements from the symbol for  $\varphi_1$ .

We get,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2\alpha_1 & x\alpha_1\alpha_2^3 & x\alpha_2 \\ x\alpha_1^2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1^2 & x\alpha_2\alpha_1\alpha_2 & x\alpha_2^2 \\ x\alpha_1^3 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2^2 \end{pmatrix}.$$

This gives rise to two more elements of  $F_{2,1}$ ,  $\varphi_3, \varphi_4$ . We see that  $\varphi_3$  is equal to  $\theta_6$ ,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2 & x\alpha_2 \\ x\alpha_1^2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2^2 \end{pmatrix},$$

and  $\varphi_4$  is equal to  $\theta_5$ ,

$$\begin{pmatrix} x\alpha_1 & x\alpha_2\alpha_1^2 & x\alpha_2\alpha_1\alpha_2 & x\alpha_2^2 \\ x\alpha_1^2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2^2 \end{pmatrix}.$$

Thus, we can represent  $\psi$  by the product  $\theta_6\theta_5\theta_4^{-1}$ .

This decomposition is obviously not unique, but there are finitely many of them with respect to the elements of depth at most 3 and the given algorithm. We could choose a different quadruple of elements from the symbol for  $\psi$  and different bases  $X$ , but there are only finitely many choices for each.

We now demonstrate a different choice of a quadruple of elements from the symbol for  $\psi$ ,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2\alpha_1 & x\alpha_1\alpha_2^3 & x\alpha_2\alpha_1 & x\alpha_2^2 \\ x\alpha_1^5 & x\alpha_1^4\alpha_2 & x\alpha_1^3\alpha_2 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix}.$$

Let

$$X = \{x\alpha_1^3, x\alpha_1^2\alpha_2, x\alpha_1\alpha_2\alpha_1, x\alpha_1\alpha_2^2, x\alpha_2\alpha_1, x\alpha_2^2\}.$$

Then, we have the following symbol for  $\psi$ ,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2\alpha_1 & x\alpha_1\alpha_2^3 & x\alpha_2\alpha_1 & x\alpha_2^2 \\ x\alpha_1^3 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2 & x\alpha_2\alpha_1 & x\alpha_2^2 \\ x\alpha_1^5 & x\alpha_1^4\alpha_2 & x\alpha_1^3\alpha_2 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix}.$$

This gives rise to elements  $\varphi_1$  and  $\varphi_2$ . We take  $\varphi_1$ , as defined by the symbol,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2 & x\alpha_2\alpha_1 & x\alpha_2^2 \\ x\alpha_1^3 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2^2 \end{pmatrix},$$

and make the reduction to

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2 & x\alpha_2 \\ x\alpha_1^3 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix}.$$

We clearly see that  $\varphi_1$  is actually  $\theta_3^{-1}$ .

Secondly, we turn our attention to  $\varphi_2$ , given by the symbol

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2 & x\alpha_2\alpha_1 & x\alpha_2^2 \\ x\alpha_1^4 & x\alpha_1^3\alpha_2 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix}.$$

We can now make an interesting choice of quadruple of elements of the symbol,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2 & x\alpha_2\alpha_1 & x\alpha_2^2 \\ x\alpha_1^4 & x\alpha_1^3\alpha_2 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix}.$$

Notice that we have an over lap of our choice of elements. Thus, we need to add two rows following the procedure in the previous sections. Let

$$X_1 = \{x\alpha_1^2, x\alpha_1\alpha_2\alpha_1, x\alpha_1\alpha_2^2, x\alpha_2\alpha_1, x\alpha_2\alpha_2\}$$

and

$$X_2 = \{x\alpha_1^3, x\alpha_1^2\alpha_2, x\alpha_1\alpha_2, x\alpha_2\alpha_1, x\alpha_2\alpha_2\}.$$

We thus get the symbol,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2 & x\alpha_2\alpha_1 & x\alpha_2^2 \\ x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2 & x\alpha_2\alpha_1 & x\alpha_2\alpha_2 \\ x\alpha_1^3 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2\alpha_2 \\ x\alpha_1^4 & x\alpha_1^3\alpha_2 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix}.$$

We define  $\varphi_3$  by the symbol,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2^2 \\ x\alpha_1^2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2\alpha_2 \end{pmatrix},$$

which defines the identity. We define  $\varphi_4$  by the symbol,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2 & x\alpha_2\alpha_1 & x\alpha_2\alpha_2 \\ x\alpha_1^3 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2\alpha_2 \end{pmatrix},$$

which reduces to the symbol,

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2\alpha_1 & x\alpha_1\alpha_2^2 & x\alpha_2 \\ x\alpha_1^3 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix},$$

which represents  $\theta_3^{-1}$ . We define  $\varphi_5$  by the symbol

$$\begin{pmatrix} x\alpha_1^2 & x\alpha_1\alpha_2 & x\alpha_2\alpha_1 & x\alpha_2\alpha_2 \\ x\alpha_1^3 & x\alpha_1^2\alpha_2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} x\alpha_1 & x\alpha_2\alpha_1 & x\alpha_2\alpha_2 \\ x\alpha_1^2 & x\alpha_1\alpha_2 & x\alpha_2 \end{pmatrix},$$

which represents the element  $\theta_1^{-1}$ .

Therefore, we have another decomposition of  $\psi$  as  $\theta_3^{-2}\theta_1^{-1}$ .

#### 4. FINITELY GENERATED SUBGROUPS OF $F_{2,1}$

Let  $H$  be a finitely generated subgroup of  $F_{2,1}$ , generated by  $\psi_1, \dots, \psi_n$ . For each decomposition  $a_{i,j_i} \in \mathbb{M}(\psi_i)$  (see Definition 3.8), with  $j_i = 1, \dots, |\mathbb{M}(\psi_i)|$ , we can form an  $n$ -tuple

$$(1) \quad (a_{1,j_1}, \dots, a_{n,j_n})$$

for each set of values  $j_1, \dots, j_n$ , giving a representation for the generating elements of  $H$  as products of elements of depth at most 3.

**Definition 4.1.** We define  $\mathbb{M}(H)$  to be the set of all possible  $n$ -tuples of the form

$$(a_{1,j_1}, \dots, a_{n,j_n})$$

with  $a_{i,j_i} \in \mathbb{M}(\psi_i)$  and  $j_i = 1, \dots, |\mathbb{M}(\psi_i)|$ .

It is clear that the set  $\mathbb{M}(H)$  is finite, as each set  $\mathbb{M}(\psi)$  is finite. For each  $n$ -tuple given by (1) we can form an abstract graph in the following way.

*Remark 4.2.* This idea is similar to the abstract graphs constructed in [13], however no foldings will occur.

**Definition 4.3.** Let  $\mathbf{a} = (a_1, \dots, a_n)$  be an  $n$ -tuple of elements of  $F_{2,1}$  such that the  $a_i$ 's are words in elements  $\theta_j, \theta_j^{-1}$  of depth at most 3 for  $i = 1, \dots, n$ .

We define the directed abstract graph  $\mathcal{G}_{\mathbf{a}}$  in the following way:

- form a bouquet of  $n$  loops oriented in the clockwise direction with central vertex  $\bigcirc$ ,
- the  $i^{\text{th}}$  loop is decorated with the word  $a_i$  in terms of elements  $\theta_j, \theta_j^{-1}$  for  $i = 1, \dots, n$ ,
- each edge labeled by  $\theta_j, \theta_j^{-1}$  in the loop  $a_i$  is oriented clockwise for  $\theta_j$  and anti-clockwise for  $\theta_j^{-1}$ .

This defines a directed abstract graph  $\mathcal{G}_{\mathbf{a}}$  related to an  $n$ -tuple  $\mathbf{a}$ .

As each element  $\mathbf{a} = (a_1, \dots, a_n)$  of  $\mathbb{M}(H)$  gives rise to exactly one abstract graph  $\mathcal{G}_{\mathbf{a}}$ , the set of all abstract graphs constructed from the set  $\mathbb{M}(H)$  is finite.

This encodes the finitely generated subgroup  $H$  as a finite set of graphs. For each graph  $\mathcal{G}_{\mathbf{a}}$ , all cycles traversed on the graph from  $\bigcirc$  to  $\bigcirc$  gives rise to elements of the subgroup  $H$  given as a decomposition into elements  $\theta_i, \theta_i^{-1}$  of depth at most 3.

*Remark 4.4.* We note that  $F_{2,1}$  does not contain a non-abelian free group of rank 2, thus can not contain a subgroup isomorphic to  $F_2 \times F_2$ .

*Question 4.5.* Could this method of decomposition of elements of Thompson's group  $F$  be used to solve the membership problem for the group?

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